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The dichotomy spectrum for random dynamical systems and pitchfork bifurcations with additive noise

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Abstract. We develop the *dichotomy spectrum* for random dynamical systems and demonstrate its use in the characterization of pitchfork bifurcations for random dynamical systems with additive noise.

Crauel and Flandoli (*J. Dynam. Differential Equations* **10** (1998) 259–274) had shown earlier that adding noise to a system with a deterministic pitchfork bifurcation yields a unique attracting random equilibrium with negative Lyapunov exponent throughout, thus "destroying" this bifurcation. Indeed, we show that in this example the dynamics before and after the underlying deterministic bifurcation point are topologically equivalent.

However, in apparent paradox to (*J. Dynam. Differential Equations* **10** (1998) 259–274), we show that there is after all a qualitative change in the random dynamics at the underlying deterministic bifurcation point, characterized by the transition from a hyperbolic to a non-hyperbolic dichotomy spectrum. This breakdown manifests itself also in the loss of uniform attractivity, a loss of experimental observability of the Lyapunov exponent, and a loss of equivalence under uniformly continuous topological conjugacies.

Résumé. Nous développons le *spectre de dichotomie* pour les systèmes dynamiques aléatoires et nous démontrons son utilité pour la caractérisation des bifurcations de fourches dans des systèmes dynamiques aléatoires avec du bruit additif.

Crauel et Flandoli (*J. Dynam. Differential Equations* **10** (1998) 259–274) ont précédemment montré que l'ajout de bruit additif à un système comprenant une bifurcation de fourche déterministe produit un unique équilibre aléatoire attractif avec un exposant de Lyapunov négatif partout, « détruisant » ainsi cette bifurcation. En effet, nous montrons dans cet exemple que la dynamique avant et après le point de bifurcation déterministe sous-jacent sont topologiquement équivalentes.

Cependant, dans un paradoxe apparent avec (*J. Dynam. Differential Equations* **10** (1998) 259–274), nous montrons qu'il y a après tout un changement qualitatif du système aléatoire au point du bifurcation déterministe sous-jacent, caractérisé par la transition du spectre de dichotomie hyperbolique à un spectre non-hyperbolique. Cette rupture se manifeste elle-même aussi dans une perte d'attractivité uniforme, une perte d'observabilité expérimentale de l'exposant de Lyapunov, et une perte d'équivalence sous conjugaisons topologiques uniformes et continues.

MSC: 37H15; 37H20

Keywords: Dichtomy spectrum; Finite-time Lyapunov exponent; Pitchfork bifurcation; Random dynamical system; Topological equivalence

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1. Introduction

Despite its importance for applications, relatively little progress has been made towards the development of a bifurcation theory for random dynamical systems. Main contributions have been made by Ludwig Arnold and co-workers [1], distinguishing between *phenomenological* (P-) and *dynamical* (D-) bifurcations. P-bifurcations refer to qualitative changes in the profile of stationary probability densities [29]. This concept carries substantial drawbacks such as providing reference only to static properties, and not being independent of the choice of coordinates. D-bifurcations refer to the bifurcation of a new invariant measure from a given invariant reference measure, in the sense of weak convergence, and are associated with a qualitative change in the Lyapunov spectrum. They have been studied mainly in the case of multiplicative noise [4,10,31], and numerically [2,19].

In this paper, we contribute to the bifurcation theory of random dynamical systems by shedding new light on the influential paper *Additive noise destroys a pitchfork bifurcation* by Crauel and Flandoli [9], in which the stochastic differential equation

$$\mathrm{d}x = \left(\alpha x - x^3\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{1.1}$$

with two-sided Wiener process $(W_t)_{t\in\mathbb{R}}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, was studied. In the deterministic (noise-free) case, $\sigma = 0$, this system has a pitchfork bifurcation of equilibria: if $\alpha < 0$ there is one equilibrium ($x = 0$) which is globally attractive, and if $\alpha > 0$, the trivial equilibrium is repulsive and there are two additional attractive equilibria $\pm\sqrt{\alpha}$. [9] establish the following facts in the presence of noise, i.e. when $|\sigma| > 0$:

(i) For all $\alpha \in \mathbb{R}$, there is a unique globally attracting random equilibrium $\{a_\alpha(\omega)\}_{\omega\in\Omega}$.
(ii) The Lyapunov exponent associated to $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is negative for all $\alpha \in \mathbb{R}$.

As a result, [9] concludes that the pitchfork bifurcation is destroyed by the additive noise. (This refers to the absence of D-bifurcation, as (1.1) admits a qualitative change P-bifurcation, see [1, p. 473].) However, we are inclined to argue that the pitchfork bifurcation is not destroyed by additive noise, on the basis of the following additional facts concerning the dynamics near the bifurcation point, that we obtain in this paper:

(i) The attracting random equilibrium $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is uniformly attractive only if $\alpha < 0$ (Theorem 4.2).
(ii) At the bifurcation point there is a change in the practical observability of the Lyapunov exponent: when $\alpha < 0$ all finite-time Lyapunov exponents are negative, but when $\alpha > 0$ there is a positive probability to observe positive finite-time Lyapunov exponents, irrespective of the length of time interval under consideration (Theorem 4.3).
(iii) The bifurcation point $\alpha = 0$ is characterized by a qualitative change in the dichotomy spectrum associated to $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ (Theorem 4.5). In addition, we show that the dichotomy spectrum is directly related to the observability range of the finite-time Lyapunov spectrum (Theorem 4.6).

In light of these findings, we thus argue for the recognition of qualitative properties of the dichotomy spectrum as an additional indicator for bifurcations of random dynamical systems. Spectral studies of random dynamical systems have focused mainly on Lyapunov exponents [1,6], but here we develop an alternative spectral theory based on exponential dichotomies that is related to the Sacker–Sell (or dichotomy) spectrum for nonautonomous differential equations. The original construction due to R. J. Sacker and G. R. Sell [27] requires a compact base set (which can be obtained, for instance, from an almost periodic differential equation). Alternative approaches to the dichotomy spectrum [3,5,25,26,28] hold in the general non-compact case, and we use similar techniques for the construction of the dichotomy spectrum by combining them with ergodic properties of the base flow. We note that the relationship between the dichotomy spectrum and Lyapunov spectrum has also been explored in [18] in the special case that the base space of a random dynamical system is a compact Hausdorff space, but our setup does not require a topological structure of the base.

In analogy to the corresponding bifurcation theory for one-dimensional deterministic dynamical systems, we finally study whether the pitchfork bifurcation with additive noise can be characterized in terms of a breakdown of topological equivalence. We recall that two random dynamical systems (θ, φ_1) and (θ, φ_2) are said to be topologically equivalent if there are families $\{h_\omega\}_{\omega\in\Omega}$ of homeomorphisms of the state space such that $\varphi_2(t, \omega, h_\omega(x)) = h_{\theta_t\omega}(\varphi_1(t, \omega, x))$, almost surely. We establish the following results for the stochastic differential equation (1.1):

(i) Throughout the bifurcation, i.e. for $|\alpha|$ sufficiently small, the resulting dynamics are topologically equivalent (Theorem 5.2).

(ii) There does not exist a uniformly continuous topological conjugacy between the dynamics of cases with positive and negative parameter α (Theorem 5.5).

These results lead us to propose the association of bifurcations of random dynamical systems with a breakdown of *uniform* topological equivalence, rather than the weaker form of general topological equivalence with no requirement on uniform continuity of the involved conjugacy. Note that uniformity of equivalence transformations plays an important role in the notion of equivalence for nonautonomous linear systems (i.e. in contrast to random systems, the base set of nonautonomous systems is not a probability but a topological space), see [23].

This paper is organised as follows. In Section 2, invariant projectors and exponential dichotomies are introduced for random dynamical systems. Section 3 is devoted to the development of the dichotomy spectrum. In Section 4, we discuss the pitchfork bifurcation with additive noise, reviewing the results of [9] and develop our main results in relationship to the dichotomy spectrum. Finally, in Section 5, we discuss the existence (and absence) of (uniform) topological equivalence of the dynamics in the neighbourhood of the bifurcation point. Important preliminaries on random dynamical systems are provided in the Appendix.

2. Exponential dichotomies for random dynamical systems

In this section, we define invariant projectors and exponential dichotomies as tools to describe hyperbolicity and (un)stable manifolds of linear random dynamical systems.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (X, d) be a metric space. A *random dynamical system* (θ, φ) (RDS for short) consists of a metric dynamical system $\theta : \mathbb{T} \times \Omega \to \Omega$, where $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ (which models the noise, see Appendix) and a $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping $\varphi : \mathbb{T} \times \Omega \times X \to X$ (which models the dynamics of the system) fulfilling

(i) $\varphi(0, \omega, x) = x$ for all $\omega \in \Omega$ and $x \in X$,
(ii) $\varphi(t + s, \omega, x) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x))$ for all $t, s \in \mathbb{T}$, $\omega \in \Omega$ and $x \in X$.

We assume throughout the document that the mapping $x \mapsto \varphi(t, \omega, x)$ is continuous for all $t \in \mathbb{T}$ and $\omega \in \Omega$. Note that we frequently use the abbreviation $\varphi(t, \omega)x$ for $\varphi(t, \omega, x)$ (even if the random dynamical system under consideration is nonlinear). We also say that a random dynamical system (θ, φ) is ergodic if θ is ergodic.

For the spectral theory part of this paper, suppose that the phase space X is given by the Euclidean space $\mathbb{R}^d$. A random dynamical system (θ, φ) is called *linear* if for given $\alpha, \beta \in \mathbb{R}$, we have

$$\varphi(t, \omega)(\alpha x + \beta y) = \alpha\varphi(t, \omega)x + \beta\varphi(t, \omega)y$$

for all $t \in \mathbb{T}$, $\omega \in \Omega$ and $x, y \in \mathbb{R}^d$. Given a linear random dynamical system (θ, φ), there exists a corresponding matrix-valued function $\Phi : \mathbb{T} \times \Omega \to \mathbb{R}^{d\times d}$ with $\Phi(t, \omega)x = \varphi(t, \omega)x$ for all $t \in \mathbb{T}$, $\omega \in \Omega$ and $x \in \mathbb{R}^d$.

Given a linear random dynamical system (θ, Φ), an invariant random set M (see Appendix) is called a *linear random set* if for each $\omega \in \Omega$, the set $M(\omega)$ is a linear subspace of $\mathbb{R}^d$. Given linear random sets M_1, M_2,

$$\omega \mapsto M_1(\omega) \cap M_2(\omega) \quad \text{and} \quad \omega \mapsto M_1(\omega) + M_2(\omega)$$

are also linear random sets, denoted by $M_1 \cap M_2$ and $M_1 + M_2$, respectively. A finite sum $M_1 + \cdots + M_n$ of linear random sets is called a *Whitney sum* $M_1 \oplus \cdots \oplus M_n$ if $M_1(\omega) \oplus \cdots \oplus M_n(\omega) = \mathbb{R}^d$ holds for almost all $\omega \in \Omega$.

An *invariant projector* of (θ, Φ) is a measurable function $P : \Omega \to \mathbb{R}^{d\times d}$ with

$$P(\omega) = P(\omega)^2 \quad \text{and} \quad P(\theta_t\omega)\Phi(t, \omega) = \Phi(t, \omega)P(\omega) \quad \text{for all } t \in \mathbb{T} \text{ and } \omega \in \Omega.$$

The *range*

$$\mathcal{R}(P) := \{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in \mathcal{R}P(\omega)\}$$

and the *null space*

$$\mathcal{N}(P) := \{(\omega, x) \in \Omega \times \mathbb{R}^d : x \in \mathcal{N}P(\omega)\}$$

of an invariant projector P are linear random sets of (θ, Φ) such that $\mathcal{R}(P) \oplus \mathcal{N}(P) = \Omega \times \mathbb{R}^d$.

The following proposition says that, provided ergodicity, the dimensions of the range and the null space of an invariant projector are almost surely constant.

Proposition 2.1. *Let $P : \Omega \to \mathbb{R}^{d\times d}$ be an invariant projector of an ergodic linear random dynamical system (θ, Φ). Then*

(i) *the mapping $\omega \mapsto \operatorname{rk} P(\omega)$ is measurable, and*
(ii) *$\operatorname{rk} P(\omega)$ is almost surely constant.*

Proof. (i) We first show that the mapping $A \mapsto \operatorname{rk} A$ on $\mathbb{R}^{d\times d}$ is lower semi-continuous. For this purpose, let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of matrices in $\mathbb{R}^{d\times d}$ which converges to $A \in \mathbb{R}^{d\times d}$, and define $r := \operatorname{rk} A$. Then there exist non-zero vectors $x_1, \dots, x_r$ such that $Ax_1, \dots, Ax_r$ are linearly independent, which implies that $\det[Ax_1, \dots, Ax_r, x_{r+1}, \dots, x_d] \neq 0$ for some vectors $x_{r+1}, \dots, x_d \in \mathbb{R}^d$. Since $\lim_{k\to\infty} A_k = A$, one gets

$$\lim_{k\to\infty} \det[A_k x_1, \dots, A_k x_r, x_{r+1}, \dots, x_d] = \det[Ax_1, \dots, Ax_r, x_{r+1}, \dots, x_d].$$

Hence, there exists a $k_0 \in \mathbb{N}$ such that vectors $A_k x_1, \dots, A_k x_r$ are linearly independent for $k \geq k_0$, and thus, $\operatorname{rk} A_k \geq r$ for all $k \geq k_0$. Consequently, the lower semi-continuity of the mapping $A \mapsto \operatorname{rk} A$ is proved. Therefore, the map $\mathbb{R}^{d\times d} \to \mathbb{N}_0$, $A \mapsto \operatorname{rk} A$ is the limit of a monotonically increasing sequence of continuous functions [30] and thus is measurable. The proof of this part is complete. (ii) By invariance of P, we get that

$$P(\theta_t\omega) = \Phi(t,\omega)P(\omega)\Phi(t,\omega)^{-1},$$

which implies that $\operatorname{rk} P(\theta_t\omega) = \operatorname{rk} P(\omega)$. This together with ergodicity of θ and measurability of the map $\omega \mapsto \operatorname{rk} P(\omega)$ as shown in (i) gives that $\operatorname{rk} P(\omega)$ is almost constant. □

According to Proposition 2.1, the rank of an invariant projector P can be defined via

$$\operatorname{rk} P := \dim \mathcal{R}(P) := \dim \mathcal{R}P(\omega) \quad \text{for almost all } \omega \in \Omega,$$

and one sets

$$\dim \mathcal{N}(P) := \dim \mathcal{N}P(\omega) \quad \text{for almost all } \omega \in \Omega.$$

The following notion of an exponential dichotomy describes uniform exponential splitting of linear random dynamical systems.

Definition 2.2 (Exponential dichotomy). Let (θ, Φ) be a linear random dynamical system, and let $\gamma \in \mathbb{R}$ and $P_\gamma : \Omega \to \mathbb{R}^{d\times d}$ be an invariant projector of (θ, Φ). Then (θ, Φ) is said to admit an *exponential dichotomy* with growth rate $\gamma \in \mathbb{R}$, constants $\alpha > 0$, $K \geq 1$ and projector P_γ if for almost all $\omega \in \Omega$, one has

$$\left\|\Phi(t,\omega)P_\gamma(\omega)\right\| \leq Ke^{(\gamma-\alpha)t} \quad \text{for all } t \geq 0,$$
$$\left\|\Phi(t,\omega)\big(\mathbb{1} - P_\gamma(\omega)\big)\right\| \leq Ke^{(\gamma+\alpha)t} \quad \text{for all } t \leq 0.$$

The following proposition shows that the ranges and null spaces of invariant projectors are given by sums of Oseledets subspaces.

Proposition 2.3. *Let (θ, Φ) be an ergodic linear random dynamical system which satisfies the integrability condition of Oseledets Multiplicative Ergodic Theorem (see Appendix). Let $\lambda_1 > \cdots > \lambda_p$ and $O_1(\omega), \dots, O_p(\omega)$ denote the Lyapunov exponents and the associated Oseledets subspaces of (θ, Φ), respectively, and suppose that Φ admits an exponential dichotomy with growth rate $\gamma \in \mathbb{R}$ and projector P_γ. Then the following statements hold:*

(i) $\gamma \notin \{\lambda_1, \dots, \lambda_p\}$.

(ii) *Define* $k := \max\{i \in \{0, \dots, p\} : \lambda_i > \gamma\}$ *with the convention that* $\lambda_0 = \infty$. *Then for almost all* $\omega \in \Omega$, *one has*

$$\mathcal{N}P_\gamma(\omega) = \bigoplus_{i=1}^{k} O_i(\omega) \quad \textit{and} \quad \mathcal{R}P_\gamma(\omega) = \bigoplus_{i=k+1}^{p} O_i(\omega).$$

Proof. (i) Suppose to the contrary that $\gamma = \lambda_k$ for some $k \in \{1, \dots, p\}$. Because of the Multiplicative Ergodic Theorem, we have

$$\lim_{t\to\infty} \frac{1}{t} \ln\left\|\Phi(t,\omega)v\right\| = \lambda_k = \gamma \quad \text{for all } v \in O_k(\omega) \setminus \{0\}. \tag{2.1}$$

On the other hand, for all $v \in \mathcal{R}P_\gamma(\omega)$ we get $\|\Phi(t,\omega)v\| \le Ke^{(\gamma-\alpha)t}\|v\|$ for all $t \ge 0$. Thus,

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left\|\Phi(t,\omega)v\right\| \le \gamma - \alpha \quad \text{for all } v \in \mathcal{R}P_\gamma(\omega),$$

which together with (2.1) implies that $O_k(\omega) \cap \mathcal{R}P_\gamma(\omega) = \{0\}$. Similarly, using the fact that

$$\lim_{t\to-\infty} \frac{1}{t} \ln\left\|\Phi(t,\omega)v\right\| = \lambda_k = \gamma \quad \text{for all } v \in O_k(\omega) \setminus \{0\}$$

and Definition 2.2, we obtain that $O_k(\omega) \cap \mathcal{N}P_\gamma(\omega) = \{0\}$. Consequently, $O_k(\omega) = \{0\}$ and it leads to a contradiction.

(ii) Let $v \in \mathcal{R}P_\gamma(\omega) \setminus \{0\}$ be arbitrary. Then, according to Definition 2.2 and the definition of k we obtain that

$$\lim_{t\to\infty} \frac{1}{t} \ln\left\|\Phi(t,\omega)v\right\| \le \gamma - \alpha < \lambda_k. \tag{2.2}$$

Now we write v in the form $v = v_i + v_{i+1} + \cdots + v_p$, where $i \in \{1, \dots, p\}$ with $v_i \neq 0$ and $v_j \in O_j(\omega)$ for all $j = i, \dots, p$. Using the fact that for $j \in \{i, \dots, p\}$ with $v_j \neq 0$

$$\lim_{t\to\infty} \frac{1}{t} \ln\left\|\Phi(t,\omega)v_j\right\| = \lambda_j \le \lambda_i,$$

we obtain that

$$\lim_{t\to\infty} \frac{1}{t} \ln\left\|\Phi(t,\omega)v\right\| = \lambda_i,$$

which together with (2.2) implies that $i \ge k+1$ and therefore $\mathcal{R}P_\gamma(\omega) \subset \bigoplus_{i=k+1}^{p} O_i(\omega)$. Similarly, we also get that $\mathcal{N}P_\gamma(\omega) \subset \bigoplus_{i=1}^{k} O_i(\omega)$. On the other hand,

$$\mathbb{R}^d = \mathcal{N}P_\gamma(\omega) \oplus \mathcal{R}P_\gamma(\omega) = \bigoplus_{i=1}^{k} O_i(\omega) \oplus \bigoplus_{i=k+1}^{p} O_i(\omega).$$

Consequently, we have $\mathcal{R}P_\gamma(\omega) = \bigoplus_{i=k+1}^{p} O_i(\omega)$ and $\mathcal{N}P_\gamma(\omega) = \bigoplus_{i=1}^{k} O_i(\omega)$. The proof is complete. □

The monotonicity of the exponential function implies the following basic criteria for the existence of exponential dichotomies.

Lemma 2.4. *Suppose that the linear random system* (θ, Φ) *admits an exponential dichotomy with growth rate* γ *and projector* P_γ. *Then the following statements are fulfilled*:

(i) *If* $P_\gamma \equiv \mathbb{1}$ *almost surely, then* (θ, Φ) *admits an exponential dichotomy with growth rate* ζ *and invariant projector* $P_\zeta \equiv \mathbb{1}$ *for all* $\zeta > \gamma$.

(ii) *If $P_\gamma \equiv 0$ almost surely, then (θ, Φ) admits an exponential dichotomy with growth rate ζ and invariant projector $P_\zeta \equiv 0$ for all $\zeta < \gamma$.*

Given $\gamma \in \mathbb{R}$, a function $g : \mathbb{R} \to \mathbb{R}^d$ is called *γ^+-exponentially bounded* if $\sup_{t \in [0,\infty)} \|g(t)\| e^{-\gamma t} < \infty$. Accordingly, one says that a function $g : \mathbb{R} \to \mathbb{R}^d$ is *γ^--exponentially bounded* if $\sup_{t \in (-\infty,0]} \|g(t)\| e^{-\gamma t} < \infty$.

We define for all $\gamma \in \mathbb{R}$

$$\mathcal{S}^\gamma := \{(\omega, x) \in \Omega \times \mathbb{R}^d : \Phi(\cdot, \omega)x \text{ is } \gamma^+\text{-exponentially bounded}\},$$

and

$$\mathcal{U}^\gamma := \{(\omega, x) \in \Omega \times \mathbb{R}^d : \Phi(\cdot, \omega)x \text{ is } \gamma^-\text{-exponentially bounded}\}.$$

It is obvious that $\mathcal{S}^\gamma$ and $\mathcal{U}^\gamma$ are linear invariant random sets of (θ, Φ), and given $\gamma \le \zeta$, the relations $\mathcal{S}^\gamma \subset \mathcal{S}^\zeta$ and $\mathcal{U}^\gamma \supset \mathcal{U}^\zeta$ are fulfilled.

The relationship between the projectors of exponential dichotomies with growth rate γ and the sets $\mathcal{S}^\gamma$ and $\mathcal{U}^\gamma$ will now be discussed.

Proposition 2.5. *If the linear random dynamical system (θ, Φ) admits an exponential dichotomy with growth rate γ and projector P_γ, then $\mathcal{N}(P_\gamma) = \mathcal{U}^\gamma$ and $\mathcal{R}(P_\gamma) = \mathcal{S}^\gamma$ almost surely.*

Proof. Suppose that (θ, Φ) admits an exponential dichotomy with growth rate γ, constants α, K and projector P_γ. This means that for almost all $\omega \in \Omega$, one has

$$\begin{aligned} &\|\Phi(t,\omega) P_\gamma(\omega)\| \le K e^{(\gamma-\alpha)t} \quad \text{for all } t \ge 0, \\ &\|\Phi(t,\omega)(\mathbb{1} - P_\gamma(\omega))\| \le K e^{(\gamma+\alpha)t} \quad \text{for all } t \le 0. \end{aligned} \tag{2.3}$$

We now prove the relation $\mathcal{N}(P_\gamma) = \mathcal{U}^\gamma$ almost surely. ($\supset$) Choose $(\omega, x) \in \mathcal{U}^\gamma$ with ω in the full measure set $F \in \mathcal{F}$ where both (2.3) and Birkhoff's Ergodic Theorem hold, and with x arbitrary. We have that $\|\Phi(t,\omega)x\| \le Ce^{\gamma t}$ for all $t \le 0$ and some real constant $C > 0$. Write $x = x_1 + x_2$ with $x_1 \in \mathcal{R}P_\gamma(\omega)$ and $x_2 \in \mathcal{N}P_\gamma(\omega)$. By Birkhoff's Ergodic Theorem there exists a sequence $t_i \to -\infty$ such that for all $i \in \mathbb{N}$ one has $\theta_{t_i}\omega \in F$, and hence

$$\begin{aligned} \|x_1\| &= \|\Phi(-t_i, \theta_{t_i}\omega)\Phi(t_i,\omega)P_\gamma(\omega)x\| \\ &= \|\Phi(-t_i, \theta_{t_i}\omega)P_\gamma(\theta_{t_i}\omega)\Phi(t_i,\omega)x\| \\ &\le K e^{-(\gamma-\alpha)t_i}\|\Phi(t_i,\omega)x\| \le CKe^{-(\gamma-\alpha)t_i}e^{\gamma t_i} = CKe^{\alpha t_i}. \end{aligned}$$

The right-hand side of this inequality converges to zero in the limit $i \to \infty$. This implies $x_1 = 0$, and thus, $(\omega, x) \in \mathcal{N}(P_\gamma)$. ($\subset$) Choose $(\omega, x) \in \mathcal{N}(P_\gamma)$. Thus, for all $t \le 0$ and almost all $\omega \in \Omega$, the relation $\|\Phi(t,\omega)x\| \le Ke^{(\gamma+\alpha)t}\|x\|$ is fulfilled. This means that $\Phi(\cdot,\omega)x$ is γ^--exponentially bounded. The proof of the statement concerning the range of the projector is treated analogously. □

3. The dichotomy spectrum for random dynamical systems

We introduce the dichotomy spectrum for random dynamical systems in this section. For the definition of the dichotomy spectra, it is crucial for which growth rates, a linear random dynamical system (θ, Φ) admits an exponential dichotomy. The growth rates $\gamma = \pm\infty$ are not excluded from our considerations; in particular, one says that (θ, Φ) admits an exponential dichotomy with growth rate ∞ if there exists a $\gamma \in \mathbb{R}$ such that (θ, Φ) admits an exponential dichotomy with growth rate γ and projector $P_\gamma \equiv \mathbb{1}$. Accordingly, one says that (θ, Φ) admits an exponential dichotomy with growth rate $-\infty$ if there exists a $\gamma \in \mathbb{R}$ such that (θ, Φ) admits an exponential dichotomy with growth rate γ and projector $P_\gamma \equiv 0$.

Definition 3.1 (Dichotomy spectrum). Consider the linear random dynamical system (θ, Φ). Then the *dichotomy spectrum* of (θ, Φ) is defined by

$$\Sigma := \big\{\gamma \in \overline{\mathbb{R}} : (\theta, \Phi) \text{ does not admit an exponential dichotomy with growth rate } \gamma\big\}.$$

The corresponding *resolvent set* is defined by $\rho := \overline{\mathbb{R}} \setminus \Sigma$.

The aim of the following lemma is to analyze the topological structure of the resolvent set.

Lemma 3.2. *Consider the resolvent set ρ of a linear random dynamical system (θ, Φ). Then $\rho \cap \mathbb{R}$ is open. More precisely, for all $\gamma \in \rho \cap \mathbb{R}$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(\gamma) \subset \rho$. Furthermore, the relation $\operatorname{rk} P_\zeta = \operatorname{rk} P_\gamma$ is (almost surely) fulfilled for all $\zeta \in B_\varepsilon(\gamma)$ and every invariant projector P_γ and P_ζ of the exponential dichotomies of (θ, Φ) with growth rates γ and ζ, respectively.*

Proof. Choose $\gamma \in \rho$ arbitrarily. Since (θ, Φ) admits an exponential dichotomy with growth rate γ, there exist an invariant projector P_γ and constants $\alpha > 0$, $K \geq 1$ such that for almost all $\omega \in \Omega$, one has

$$2\big\|\Phi(t,\omega)P_\gamma(\omega)\big\| \leq Ke^{(\gamma-\alpha)t} \quad \text{for all } t \geq 0,$$
$$\big\|\Phi(t,\omega)\big(\mathbb{1} - P_\gamma(\omega)\big)\big\| \leq Ke^{(\gamma+\alpha)t} \quad \text{for all } t \leq 0.$$

Set $\varepsilon := \frac{1}{2}\alpha$, and choose $\zeta \in B_\varepsilon(\gamma)$. It follows that for almost all $\omega \in \Omega$,

$$2\big\|\Phi(t,\omega)P_\gamma(\omega)\big\| \leq Ke^{(\zeta-\frac{\alpha}{2})t} \quad \text{for all } t \geq 0,$$
$$\big\|\Phi(t,\omega)\big(\mathbb{1} - P_\gamma(\omega)\big)\big\| \leq Ke^{(\zeta+\frac{\alpha}{2})t} \quad \text{for all } t \leq 0.$$

This yields $\zeta \in \rho$, and it follows that $\operatorname{rk} P_\zeta = \operatorname{rk} P_\gamma$ for any projector P_ζ of the exponential dichotomy with growth rate ζ. This finishes the proof of this lemma. □

Lemma 3.3. *Consider the resolvent set ρ of a linear random dynamical system (θ, Φ), and let $\gamma_1, \gamma_2 \in \rho \cap \mathbb{R}$ such that $\gamma_1 < \gamma_2$. Moreover, choose invariant projectors P_{γ_1} and P_{γ_2} for the corresponding exponential dichotomies with growth rates γ_1 and γ_2. Then the relation $\operatorname{rk} P_{\gamma_1} \leq \operatorname{rk} P_{\gamma_2}$ holds. In addition, $[\gamma_1, \gamma_2] \subset \rho$ is fulfilled if and only if $\operatorname{rk} P_{\gamma_1} = \operatorname{rk} P_{\gamma_2}$, and in this case one has that $P_\gamma = P_\zeta$ almost surely for all $\gamma, \zeta \in [\gamma_1, \gamma_2]$.*

Proof. The relation $\operatorname{rk} P_{\gamma_1} \leq \operatorname{rk} P_{\gamma_2}$ is a direct consequence of Proposition 2.5, since $\mathcal{S}^{\gamma_1} \subset \mathcal{S}^{\gamma_2}$ and $\mathcal{U}^{\gamma_1} \supset \mathcal{U}^{\gamma_2}$. Now assume that $[\gamma_1, \gamma_2] \subset \rho$. Arguing contrapositively, suppose that $\operatorname{rk} P_{\gamma_1} \neq \operatorname{rk} P_{\gamma_2}$, and choose invariant projectors P_ζ, $\zeta \in (\gamma_1, \gamma_2)$, for the exponential dichotomies of (θ, Φ) with growth rate ζ. Define

$$\zeta_0 := \sup\big\{\zeta \in [\gamma_1, \gamma_2] : \operatorname{rk} P_\zeta \neq \operatorname{rk} P_{\gamma_2}\big\}.$$

Due to Lemma 3.2, there exists an $\varepsilon > 0$ such that $\operatorname{rk} P_{\zeta_0} = \operatorname{rk} P_\zeta$ for all $\zeta \in B_\varepsilon(\zeta_0)$. This is a contradiction to the definition of ζ_0. Conversely, let $\operatorname{rk} P_{\gamma_1} = \operatorname{rk} P_{\gamma_2}$, then Proposition 2.5 together with the fact that $\mathcal{S}^{\gamma_1} \subset \mathcal{S}^{\gamma_2}$ and $\mathcal{U}^{\gamma_1} \supset \mathcal{U}^{\gamma_2}$ yields that $\mathcal{R}(P_{\gamma_1}) = \mathcal{R}(P_{\gamma_2})$ and $\mathcal{N}(P_{\gamma_1}) = \mathcal{N}(P_{\gamma_2})$ almost surely, hence $P_{\gamma_1} = P_{\gamma_2}$ almost surely and P_{γ_2} is also an invariant projector of the exponential dichotomy with growth rate γ_1. Thus, one obtains for almost all $\omega \in \Omega$,

$$\big\|\Phi(t,\omega)P_{\gamma_2}(\omega)\big\| \leq K_1 e^{(\gamma_1-\alpha_1)t} \quad \text{for all } t \geq 0$$

for some $K_1 \geq 1$ and $\alpha_1 > 0$, and

$$\big\|\Phi(t,\omega)\big(\mathbb{1} - P_{\gamma_2}(\omega)\big)\big\| \leq K_2 e^{(\gamma_2+\alpha_2)t} \quad \text{for all } t \leq 0$$

with some $K_2 \geq 1$ and $\alpha_2 > 0$. For all $\gamma \in [\gamma_1, \gamma_2]$ these two inequalities imply, by setting $K := \max\{K_1, K_2\}$ and $\alpha := \min\{\alpha_1, \alpha_2\}$, that for almost all $\omega \in \Omega$

$$2\left\|\Phi(t,\omega)P_{\gamma_2}(\omega)\right\| \leq Ke^{(\gamma-\alpha)t} \quad \text{for all } t \geq 0,$$
$$\left\|\Phi(t,\omega)\big(\mathbb{1} - P_{\gamma_2}(\omega)\big)\right\| \leq Ke^{(\gamma+\alpha)t} \quad \text{for all } t \leq 0.$$

This means that $\gamma \in \rho$, and thus, $[\gamma_1, \gamma_2] \subset \rho$. Now for arbitrary $\gamma, \zeta \in [\gamma_1, \gamma_2]$ with $\gamma \leq \zeta$ one has $\operatorname{rk} P_\gamma \leq \operatorname{rk} P_\zeta$, and since the relation $\operatorname{rk} P_{\gamma_1} = \operatorname{rk} P_{\gamma_2}$ also holds, one must have that $\operatorname{rk} P_\gamma = \operatorname{rk} P_\zeta$. Then Proposition 2.5 together with the fact that $\mathcal{S}^\gamma \subset \mathcal{S}^\zeta$ and $\mathcal{U}^\gamma \supset \mathcal{U}^\zeta$ yields that $\mathcal{R}(P_\gamma) = \mathcal{R}(P_\zeta)$ and $\mathcal{N}(P_\gamma) = \mathcal{N}(P_\zeta)$ almost surely, and hence $P_\gamma = P_\zeta$ almost surely. □

For an arbitrarily chosen $a \in \mathbb{R}$, define

$$[-\infty, a] := (-\infty, a] \cup \{-\infty\}, \qquad [a, \infty] := [a, \infty) \cup \{\infty\}$$

and

$$[-\infty, -\infty] := \{-\infty\}, \qquad [\infty, \infty] := \{\infty\}, \qquad [-\infty, \infty] := \overline{\mathbb{R}}.$$

The following *Spectral Theorem*, describes that the dichotomy spectrum consists of at least one and at most d closed intervals.

Theorem 3.4 (Spectral Theorem). *Let (θ, Φ) be a linear random dynamical system with dichotomy spectrum Σ. Then there exists an $n \in \{1, \dots, d\}$ such that*

$$\Sigma = [a_1, b_1] \cup \cdots \cup [a_n, b_n]$$

with $-\infty \leq a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n \leq \infty$.

Proof. Due to Lemma 3.2, the resolvent set $\rho \cap \mathbb{R}$ is open. Thus, $\Sigma \cap \mathbb{R}$ is the disjoint union of closed intervals. The relation $(-\infty, b_1] \subset \Sigma$ implies $[-\infty, b_1] \subset \Sigma$, because the assumption of the existence of a $\gamma \in \mathbb{R}$ such that (θ, Φ) admits an exponential dichotomy with growth rate γ and projector $P_\gamma \equiv 0$ leads to $(-\infty, \gamma] \subset \rho$ using Lemma 2.4, and this is a contradiction. Analogously, it follows from $[a_n, \infty) \subset \Sigma$ that $[a_n, \infty] \subset \Sigma$. To show the relation $n \leq d$, assume to the contrary that $n \geq d + 1$. Thus, there exist

$$\zeta_1 < \zeta_2 < \cdots < \zeta_d \in \rho$$

such that the $d + 1$ intervals $(-\infty, \zeta_1), (\zeta_1, \zeta_2), \dots, (\zeta_d, \infty)$ have nonempty intersection with the spectrum Σ. It follows from Lemma 3.3 that

$$0 \leq \operatorname{rk} P_{\zeta_1} < \operatorname{rk} P_{\zeta_2} < \cdots < \operatorname{rk} P_{\zeta_d} \leq d$$

is fulfilled for invariant projectors P_{ζ_i} of the exponential dichotomy with growth rate ζ_i, $i \in \{1, \dots, n\}$. This implies either $\operatorname{rk} P_{\zeta_1} = 0$ or $\operatorname{rk} P_{\zeta_d} = d$. Thus, either

$$[-\infty, \zeta_1] \cap \Sigma = \varnothing \quad \text{or} \quad [\zeta_d, \infty] \cap \Sigma = \varnothing$$

is fulfilled, and this is a contradiction. To show $n \geq 1$, assume that $\Sigma = \varnothing$. This implies $\{-\infty, \infty\} \subset \rho$. Thus, there exist $\zeta_1, \zeta_2 \in \mathbb{R}$ such that (θ, Φ) admits an exponential dichotomy with growth rate ζ_1 and projector $P_{\zeta_1} \equiv 0$ and an exponential dichotomy with growth rate ζ_2 and projector $P_{\zeta_2} \equiv \mathbb{1}$. Applying Lemma 3.3, one gets $(\zeta_1, \zeta_2) \cap \Sigma \neq \varnothing$. This contradiction yields $n \geq 1$ and finishes the proof of the theorem. □

Each spectral interval is associated to a so-called spectral manifold, which generalises the stable and unstable manifolds obtained by the ranges and null spaces of invariant projectors of exponential dichotomies.

Theorem 3.5 (Spectral manifolds). *Consider the dichotomy spectrum*

$$\Sigma = [a_1, b_1] \cup \cdots \cup [a_n, b_n]$$

of the linear random dynamical system (θ, Φ) *and define the invariant projectors* $P_{\gamma_0} := 0$, $P_{\gamma_n} := \mathbb{1}$, *and for* $i \in \{1, \ldots, n-1\}$, *choose* $\gamma_i \in (b_i, a_{i+1})$ *and projectors* P_{γ_i} *of the exponential dichotomy of* (θ, Φ) *with growth rate* γ_i. *Then the sets*

$$\mathcal{W}_i := \mathcal{R}(P_{\gamma_i}) \cap \mathcal{N}(P_{\gamma_{i-1}}) \quad \text{for all } i \in \{1, \ldots, n\}$$

are fiber-wise linear subsets of $\mathbb{R}^d$, *the so-called* spectral manifolds, *which form a Whitney sum, i.e. for almost all* $\omega \in \Omega$

$$\mathcal{W}_1(\omega) \oplus \cdots \oplus \mathcal{W}_n(\omega) = \mathbb{R}^d,$$

and moreover, for almost all $\omega \in \Omega$, $\mathcal{W}_i(\omega) \neq \{0\}$ *for* $i \in \{1, \ldots, n\}$.

Proof. The sets $\mathcal{W}_1, \ldots, \mathcal{W}_n$ obviously have linear fibers. We first show that $\mathcal{W}_i(\omega) \neq \{0\}$ almost surely for all $i \in \{1, \ldots, n\}$. If $\mathcal{W}_1(\omega) \neq \{0\}$ does not hold almost surely, then Proposition 2.1 implies that $P_{\gamma_1}(\omega) = 0$ almost surely, and Lemma 2.4 implies $[-\infty, \gamma_1] \cap \Sigma = \varnothing$, which is a contradiction. A similar argument may be used for $\mathcal{W}_n$. In the case $1 < i < n$, using Lemma 3.3, one obtains

$$\dim \mathcal{W}_i = \dim\bigl(\mathcal{R}(P_{\gamma_i}) \cap \mathcal{N}(P_{\gamma_{i-1}})\bigr) = \operatorname{rk} P_{\gamma_i} + d - \operatorname{rk} P_{\gamma_{i-1}} - \dim\bigl(\mathcal{R}(P_{\gamma_i}) + \mathcal{N}(P_{\gamma_{i-1}})\bigr) \geq 1.$$

Now the relation $\mathcal{W}_1(\omega) \oplus \cdots \oplus \mathcal{W}_n(\omega) = \mathbb{R}^d$ $\mathbb{P}$-a.s. will be proved. For $1 \leq i < j \leq n$, due to Proposition 2.5, the relations $\mathcal{W}_i \subset \mathcal{R}(P_{\gamma_i})$ and $\mathcal{W}_j \subset \mathcal{N}(P_{\gamma_{j-1}}) \subset \mathcal{N}(P_{\gamma_i})$ are almost surely fulfilled. This yields that, almost surely,

$$\mathcal{W}_i(\omega) \cap \mathcal{W}_j(\omega) \subset \mathcal{R}\bigl(P_{\gamma_i}(\omega)\bigr) \cap \mathcal{N}\bigl(P_{\gamma_i}(\omega)\bigr) = \{0\}.$$

One also obtains

$$\begin{aligned}
\mathbb{R}^d &= \mathcal{W}_1(\omega) + \mathcal{N}\bigl(P_{\gamma_1}(\omega)\bigr) \\
&= \mathcal{W}_1(\omega) + \mathcal{N}\bigl(P_{\gamma_1}(\omega)\bigr) \cap \bigl(\mathcal{R}\bigl(P_{\gamma_2}(\omega)\bigr) + \mathcal{N}\bigl(P_{\gamma_2}(\omega)\bigr)\bigr) \\
&= \mathcal{W}_1(\omega) + \mathcal{N}\bigl(P_{\gamma_1}(\omega)\bigr) \cap \mathcal{R}\bigl(P_{\gamma_2}(\omega)\bigr) + \mathcal{N}\bigl(P_{\gamma_2}(\omega)\bigr) = \mathcal{W}_1(\omega) + \mathcal{W}_2(\omega) + \mathcal{N}\bigl(P_{\gamma_2}(\omega)\bigr)
\end{aligned}$$

using the fact that for linear subspaces $E, F, G \subset \mathbb{R}^d$ with $E \supset G$ fulfill $E \cap (F + G) = (E \cap F) + G$. It follows inductively that

$$\mathbb{R}^d = \mathcal{W}_1(\omega) + \cdots + \mathcal{W}_n(\omega) + \mathcal{N}\bigl(P_{\gamma_n}(\omega)\bigr) = \mathcal{W}_1(\omega) + \cdots + \mathcal{W}_n(\omega)$$

for almost all $\omega \in \Omega$. □

Remark 3.6. If the linear random dynamical system (θ, Φ) under consideration fulfills the conditions of the Multiplicative Ergodic Theorem, then Proposition 2.3 implies that the spectral manifolds $\mathcal{W}_i$ of the above theorem are given by Whitney sums of Oseledets subspaces.

The remaining part of this section on the dichotomy spectrum will be devoted to the study of boundedness properties of the spectrum. Firstly, a criterion for boundedness from above and below is provided by the following proposition.

Proposition 3.7. *Consider a linear random dynamical system* (θ, Φ), *let* Σ *denote the dichotomy spectrum of* (θ, Φ), *and define*

$$\alpha^{\pm}(\omega) := \begin{cases} \ln^+(\|\Phi(1, \omega)^{\pm 1}\|), & \mathbb{T} = \mathbb{Z}, \\ \ln^+(\sup_{t \in [0,1]} \|\Phi(t, \omega)^{\pm 1}\|), & \mathbb{T} = \mathbb{R}. \end{cases}$$

Then Σ is bounded from above if and only if

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \alpha^+(\omega) < \infty,$$

and Σ is bounded from below if and only if

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \alpha^-(\omega) < \infty.$$

Consequently, if the dichotomy spectrum Σ is bounded, then Φ satisfies the integrability condition of the Multiplicative Ergodic Theorem.

Proof. Suppose that Σ is bounded from above. Then there exist $K \geq 1$ and $\Gamma \in \mathbb{R}$ such that for almost all $\omega \in \Omega$

$$\|\Phi(t,\omega)\| \leq K e^{\Gamma t} \quad \text{for all } t \geq 0,$$

which implies that $\operatorname{ess\,sup}_{\omega\in\Omega} \alpha^+(\omega) \leq \ln(K) + |\Gamma|$. On the other hand, suppose that $\operatorname{ess\,sup}_{\omega\in\Omega} \alpha^+(\omega) < \infty$. Then there exists a full measure set $F \in \mathcal{F}$ such that for all $\omega \in F$ we have $\alpha^+(\omega) \leq \beta$ for some positive number β. Define

$$\widetilde{\Omega} := \bigcap_{n\in\mathbb{Z}} \theta_n F.$$

Due to the measure preserving property of θ, we get that $\mathbb{P}(\widetilde{\Omega}) = 1$. Then for all $\omega \in \widetilde{\Omega}$, we have

$$\|\Phi(t,\omega)\| \leq \|\Phi(t-\lfloor t\rfloor, \theta_{\lfloor t\rfloor}\omega)\| \|\Phi(1,\theta_{\lfloor t\rfloor-1}\omega)\| \cdots \|\Phi(1,\omega)\| \leq e^{\beta(t+1)} \quad \text{for all } t \geq 0.$$

Let $\gamma > \beta$ be arbitrary and $\varepsilon < \gamma - \beta$. Then

$$\|\Phi(t,\omega)\| \leq e^{\beta} e^{(\gamma-\varepsilon)t} \quad \text{for all } t \geq 0,$$

which implies that Φ admits an exponential dichotomy with growth rate γ and projector $P_\gamma \equiv \mathbb{1}$, and hence $\Sigma \subset [-\infty, \beta]$. Similarly, we get that Σ is bounded from below if and only if $\operatorname{ess\,sup}_{\omega\in\Omega} \alpha^-(\omega) < \infty$. This finishes the proof of this proposition. □

The following example shows that there exist linear random dynamical systems which satisfy the integrability condition of the Multiplicative Ergodic Theorem, but which have no bounded dichotomy spectrum.

Example 3.8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space and $\theta : \mathbb{Z} \times \Omega \to \Omega$ be a metric dynamical system which is ergodic. Then there exists, by using [14, Lemma 2, p. 71], a measurable set U of the form

$$U = \bigcup_{k=1}^{\infty} \bigcup_{j=0}^{k} \theta_j U_k, \tag{3.1}$$

where U_i, $i \in \mathbb{N}$, are measurable sets such that

(i) for all $k, \ell \in \mathbb{N}$, $i \in \{0,\dots,k\}$ and $j \in \{0,\dots,\ell\}$, we have

$$\theta_j U_k \cap \theta_i U_\ell = \varnothing \quad \text{whenever } k \neq \ell \text{ or } i \neq j,$$

(ii) $0 < \mathbb{P}(U_k) \leq \frac{1}{k^3}$ for all $k \in \mathbb{N}$.

We now define a random variable $a : \Omega \to \mathbb{R}$ by

$$a(\omega) := \begin{cases} 1, & \omega \in \Omega \setminus U, \\ k, & k \text{ is even and } \omega \in \theta_j U_k, \\ \frac{1}{k}, & k \text{ is odd and } \omega \in \theta_j U_k, \end{cases}$$

with $j \in \{0, \dots, k\}$. Using the random variable a, we define a discrete-time scalar linear random dynamical system $\Phi : \mathbb{Z} \times \Omega \to \mathbb{R}$ by

$$\Phi(n, \omega) = \begin{cases} a(\theta_{n-1}\omega) \cdots a(\omega), & n \geq 1, \\ 1, & n = 0, \\ a(\theta_{-1}\omega)^{-1} \cdots a(\theta_n\omega)^{-1}, & n \leq -1. \end{cases}$$

A direct computation yields that

$$\mathbb{E} \ln^+\left(\left\|\Phi(1, \omega)\right\|\right) = \sum_{k=1}^{\infty} (2k+1)\mathbb{P}(U_{2k}) \ln(2k) \leq \sum_{k=1}^{\infty} (2k+1) \frac{\ln(2k)}{8k^3} < \infty,$$

and

$$\begin{aligned} \mathbb{E} \ln^+\left(\left\|\Phi(1, \omega)^{-1}\right\|\right) &= \sum_{k=1}^{\infty} (2k+2)\mathbb{P}(U_{2k+1}) \ln(2k+1) \\ &\leq \sum_{k=1}^{\infty} (2k+2) \frac{\ln(2k+1)}{(2k+1)^3} < \infty. \end{aligned}$$

Then the linear system Φ satisfies the integrability condition of the Multiplicative Ergodic Theorem. The fact that the dichotomy spectrum of Φ is unbounded from above follows from

$$\left\|\Phi(n, \omega)\right\| = k^n \quad \text{for all } \omega \in U_k \text{ with } k \text{ even and } 0 \leq n \leq k+1.$$

Similarly, one can prove that the spectrum is unbounded from below.

4. Random pitchfork bifurcation

We first review in Section 4.1 the main results of [9], which concern the one-dimensional stochastic differential equation

$$\mathrm{d}x = \left(\alpha x - x^3\right) \mathrm{d}t + \sigma \, \mathrm{d}W_t, \tag{4.1}$$

depending on real parameters α and σ and driven by a two-sided Wiener process $(W_t)_{t \in \mathbb{R}}$. This stochastic differential equation has a unique random equilibrium $\{a_\alpha(\omega)\}_{\omega \in \Omega}$ for all $\alpha \in \mathbb{R}$. We then show in Section 4.2 that there is a qualitative change in the random dynamics at the bifurcation point $\alpha = 0$ in the sense that after the bifurcation, the attracting random equilibria $\{a_\alpha(\omega)\}_{\omega \in \Omega}$ have qualitatively different properties for $\alpha < 0$ and $\alpha \geq 0$ with respect to uniform attraction, which is lost at the bifurcation point. We also associate this bifurcation in Section 4.3 with non-hyperbolicity of the dichotomy spectrum of the linearization at the bifurcation point.

4.1. *Existence of a unique attracting random equilibrium*

Consider the stochastic differential equation (4.1). We first look at the deterministic case $\sigma = 0$. Then for $\alpha < 0$, the ordinary differential equation (4.1) has one equilibrium ($x = 0$) which is globally attractive. For positive α, the trivial equilibrium becomes repulsive, and there are two additional equilibria, given by $\pm\sqrt{\alpha}$, which are attractive. This also means that the global attractor K_α of the deterministic equation undergoes a bifurcation from a trivial to a nontrivial object. It is given by

$$K_\alpha := \begin{cases} \{0\}, & \alpha \leq 0, \\ [-\sqrt{\alpha}, \sqrt{\alpha}], & \alpha > 0. \end{cases}$$

It was shown in [9] that such an attractor bifurcation does not persist for random attractors of the randomly perturbed system where $|\sigma| > 0$, and we will explain the details now.

The stochastic differential equation (4.1) generates a one-sided random dynamical system $(\theta : \mathbb{R} \times \Omega \to \Omega, \varphi : \mathbb{R}_0^+ \times \Omega \times \mathbb{R} \to \mathbb{R})$ which induces a Markov semigroup with transition probabilities $T(x, B)$ for $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$. Note that since solutions of (4.1) explode in backward time, the system is only defined for nonnegative times.

The stochastic differential equation (4.1) induces a Markov semigroup with transition probabilities $T(x, B)$ for $x \in \mathbb{R}$ and $B \in \mathcal{B}(\mathbb{R})$. A probability measure ρ on $\mathcal{B}(X)$ is called a *stationary measure* for the Markov semigroup if

$$\rho(B) = \int_{\mathbb{R}} T(x, B)\, \mathrm{d}\rho(x) \quad \text{for all } B \in \mathcal{B}(\mathbb{R}).$$

It can be shown [1, p. 474] that for any $\alpha \in \mathbb{R}$ and $|\sigma| > 0$, the Markov semigroup associated with (4.1) admits a unique stationary measure $\rho_{\alpha,\sigma}$ with density

$$p_{\alpha,\sigma}(x) = N_{\alpha,\sigma} \exp\left(\frac{1}{\sigma^2}\left(\alpha x^2 - \frac{1}{2}x^4\right)\right), \tag{4.2}$$

where $N_{\alpha,\sigma}$ is a normalization constant. This stationary measure corresponds to an invariant measure μ of the random dynamical system (θ, φ) generated by (4.1). The invariant measure μ has the disintegration given by

$$\mu_\omega = \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\rho \quad \text{for almost all } \omega \in \Omega.$$

It was shown in [9] that μ_ω is a Dirac measure concentrated on $a_\alpha(\omega)$, and linearizing along this invariant measure μ yields a negative Lyapunov exponent, given by

$$\lambda_\alpha = -\frac{2}{\sigma^2} \int_{\mathbb{R}} \left(\alpha x - x^3\right)^2 p_{\alpha,\sigma}(x)\, \mathrm{d}x.$$

Moreover, the family $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is the global random attractor (see Appendix), which implies that the attractor bifurcation associated with a deterministic pitchfork bifurcation (that is, K_α bifurcates from a singleton to a non-trivial object) is destroyed by noise, and μ is the unique invariant measure for the random dynamical system. Also $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is the only solution to (θ, φ) which does not explode and exists for all times.

4.2. *Qualitative changes in uniform attractivity*

In order to establish qualitative changes in the attractivity of the unique attracting random equilibrium $\{a_\alpha(\omega)\}_{\omega\in\Omega}$, a detailed understanding about the location of this attractor is needed.

Proposition 4.1. *Consider (4.1) for $\alpha \in \mathbb{R}$, and let $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ be its unique random equilibrium. Then for any $\varepsilon > 0$ and $T \geq 0$, there exists a measurable set $\mathcal{A} \in \mathcal{F}_{-\infty}^T$ (see Appendix) of positive measure such that*

$$a_\alpha(\theta_s\omega) \in (-\varepsilon, \varepsilon) \quad \textit{for all } s \in [0, T] \textit{ and } \omega \in \mathcal{A}.$$

Proof. The unique stationary measure $\rho_{\alpha,\sigma}$ for the Markov semigroup associated to (4.1) with $|\sigma| > 0$ is equivalent to the Lebesgue measure with the density function given by (4.2). The invariant measure $\delta_{a(\omega)}$ and stationary measure ρ are in correspondence by the following relations: the invariant measure is obtained as the limit of the pullback images of the stationary measure, i.e.

$$\delta_{a(\omega)} = \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\rho \quad \text{for almost all } \omega \in \Omega,$$

and the stationary measure is obtained as the expectation of the invariant measure, i.e.

$$\rho(\cdot) = \int_\Omega \delta_{a(\omega)}(\cdot)\, \mathrm{d}\mathbb{P}(\omega) \tag{4.3}$$

(see [9]). Now define

$$\eta := \frac{\varepsilon e^{-|\alpha| T}}{2(1+|\sigma|)}.$$

Since the support of ρ is the entire real line, it follows from (4.3) that the set

$$A_1 := \{\omega \in \Omega: a_\alpha(\omega) \in (-\eta, \eta)\} \tag{4.4}$$

has positive probability for any $\alpha \in \mathbb{R}$. The global pullback attractor $\{a_\alpha(\omega)\}$ is measurable with respect to the past of the noise $\mathcal{F}_{-\infty}^0$ (again see [9]), and hence $A_1 \in \mathcal{F}_{-\infty}^0$. Define

$$A_2 := \left\{\omega \in \Omega: \sup_{t\in[0,T]} |\omega(t)| \le \frac{\eta}{2}\right\} \in \mathcal{F}_0^T$$

which, by [15, Section 6.8], has positive probability. Since the sets A_1 and A_2 are independent, the set $\mathcal{A} := A_1 \cap A_2 \in \mathcal{F}_{-\infty}^T$ also has positive probability. Choose and fix an arbitrary $\omega \in \mathcal{A}$. By the definition of A_1 we have that $|a_\alpha(\omega)| < \eta$. Since $a_\alpha(\omega)$ is a random equilibrium of φ it follows, using the integral from of (4.1), that

$$a_\alpha(\theta_t\omega) = a_\alpha(\theta_s\omega) + \int_s^t \left(\alpha a_\alpha(\theta_r\omega) - a_\alpha(\theta_r\omega)^3\right) \mathrm{d}r + \sigma\left(\omega(t) - \omega(s)\right). \tag{4.5}$$

Choose and fix an arbitrary $t \in [0, T]$. Define $\mathcal{I} := \{s \in [0, t]: a_\alpha(\theta_s\omega) = 0\}$; by continuity the set $\mathcal{I}$ is closed (but possibly empty). We consider the following three cases:

Case 1. If $t \in \mathcal{I}$, then $|a_\alpha(\theta_t\omega)| = 0$.

Case 2. If $\mathcal{I}$ is not empty and $t \notin \mathcal{I}$, then $s := \sup\mathcal{I} < t$ and $a_\alpha(\theta_s\omega) = 0$. By the definition of $\mathcal{I}$ and continuity, we have either $a_\alpha(\theta_r\omega) > 0$ for all $r \in (s, t]$ or $a_\alpha(\theta_r\omega) < 0$ for all $r \in (s, t]$. Using this observation and (4.5), we obtain that

$$|a_\alpha(\theta_t\omega)| \le |\sigma|\eta + \int_s^t |\alpha| |a_\alpha(\theta_r\omega)| \,\mathrm{d}r.$$

Case 3. If $\mathcal{I}$ is empty, then either $a_\alpha(\theta_s\omega) > 0$ for all $s \in [0, t]$ or $a_\alpha(\theta_s\omega) < 0$ for all $s \in [0, t]$. Using (4.5) and noting that $|a_\alpha(\omega)| < \eta$, we arrive at the following inequality:

$$|a_\alpha(\theta_t\omega)| \le \left(1 + |\sigma|\right)\eta + \int_0^t |\alpha| |a_\alpha(\theta_s\omega)| \,\mathrm{d}s.$$

In view of the three cases above, we have that

$$|a_\alpha(\theta_t\omega)| \le \left(1 + |\sigma|\right)\eta + \int_0^t |\alpha| |a_\alpha(\theta_s\omega)| \,\mathrm{d}s \quad \text{for all } t \in [0, T].$$

Then, using Gronwall's inequality, we obtain that

$$|a_\alpha(\theta_t\omega)| \le \left(1 + |\sigma|\right)\eta e^{|\alpha| t} < \varepsilon \quad \text{for all } t \in [0, T].$$

Thus we have that for all $\omega \in \mathcal{A}$, $a_\alpha(\theta_t\omega) \in (-\varepsilon, \varepsilon)$ for all $t \in [0, T]$, which completes the proof. □

We now give a detailed description of the random bifurcation scenario for the stochastic differential equation (4.1) by means of both *asymptotic* and *finite-time* dynamical behaviour. The asymptotic description implies that there is a qualitative change in the uniformity of attraction of the unique random attractor $\{a_\alpha(\omega)\}_{\omega\in\Omega}$. On the other hand, the finite-time description shows that after the bifurcation, even if the time interval is very large, the (asymptotic) Lyapunov exponent cannot be observed with non-vanishing probability (by a finite-time Lyapunov exponent); however, before the bifurcation, the (asymptotic) Lyapunov exponent can be approximated by the finite-time Lyapunov exponent. Finite-time Lyapunov exponents for random dynamical systems have not been considered in the literature so far, but play an important role in the description of Lagrangian Coherent Structures in fluid dynamics [13].

Let $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ denote the unique random equilibrium of the stochastic differential equation (4.1). Then $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ is called *locally uniformly attractive* if there exists $\delta > 0$ such that

$$\lim_{t\to\infty} \sup_{x\in(-\delta,\delta)} \operatorname*{ess\,sup}_{\omega\in\Omega} \bigl|\varphi(t,\omega)\bigl(a_\alpha(\omega)+x\bigr) - a_\alpha(\theta_t\omega)\bigr| = 0.$$

Theorem 4.2 (Random pitchfork bifurcation, asymptotic description). *Consider the stochastic differential equation* (4.1) *with the unique attracting random equilibrium* $\{a_\alpha(\omega)\}_{\omega\in\Omega}$. *Then the following statements hold*:

(i) *For* $\alpha < 0$, *the random attractor* $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ *is locally uniformly attractive*; *in fact*, *it is even globally uniformly exponential attractive*, *i.e.*

$$\bigl|\varphi(t,\omega,x) - \varphi\bigl(t,\omega,a_\alpha(\omega)\bigr)\bigr| \le e^{\alpha t}\bigl|x - a_\alpha(\omega)\bigr| \quad \textit{for all } x\in\mathbb{R}. \tag{4.6}$$

(ii) *For* $\alpha > 0$, *the random attractor* $\{a_\alpha(\omega)\}_{\omega\in\Omega}$ *is not locally uniformly attractive.*

Proof. (i) Let $x\in\mathbb{R}$ be arbitrary such that $x \ne a_\alpha(\omega)$. Using the monotonicity of solutions, we may assume that $\varphi(t,\omega,x) > \varphi(t,\omega,a_\alpha(\omega))$ for all $t \ge 0$. The integral form of (4.1),

$$\varphi(t,\omega)x = x + \int_0^t \Bigl(\alpha\varphi(s,\omega)x - \bigl(\varphi(s,\omega)x\bigr)^3\Bigr)\,\mathrm{d}s + \sigma\omega(t)$$

yields that

$$\varphi(t,\omega)x - \varphi(t,\omega)a_\alpha(\omega) \le x - a_\alpha(\omega) + \alpha\int_0^t \bigl(\varphi(s,\omega)x - \varphi(s,\omega)a_\alpha(\omega)\bigr)\,\mathrm{d}s.$$

Using Gronwall's inequality implies (4.6), which finishes this part of the proof.

(ii) Suppose to the contrary that there exists $\delta > 0$ such that

$$\lim_{t\to\infty} \sup_{x\in(-\delta,\delta)} \operatorname*{ess\,sup}_{\omega\in\Omega} \bigl|\varphi\bigl(t,\omega,a_\alpha(\omega)+x\bigr) - a_\alpha(\theta_t\omega)\bigr| = 0,$$

which implies that there exists $N\in\mathbb{N}$ such that

$$\sup_{x\in(-\delta,\delta)} \operatorname*{ess\,sup}_{\omega\in\Omega} \bigl|\varphi\bigl(t,\omega,a_\alpha(\omega)+x\bigr) - a_\alpha(\theta_t\omega)\bigr| < \frac{\sqrt{\alpha}}{4} \quad \text{for all } t \ge N. \tag{4.7}$$

According to Proposition 4.1, there exists $\mathcal{A}\in\mathcal{F}_{-\infty}^0$ of positive probability such that $a_\alpha(\omega)\in(-\frac{\delta}{2},\frac{\delta}{2})$. Note that $-\sqrt{\alpha}$ and $\sqrt{\alpha}$ are two attractive equilibria for the deterministic differential equation

$$\dot{x} = \alpha x - x^3.$$

Let $\phi(\cdot,x_0)$ denote the solution of the above deterministic equation which satisfies $x(0)=x_0$. Then there exists $T > N$ such that

$$\phi(T,\delta/2) > \frac{\sqrt{\alpha}}{2} \quad \text{and} \quad \phi(T,-\delta/2) < -\frac{\sqrt{\alpha}}{2}. \tag{4.8}$$

For any $\varepsilon > 0$, we define

$$\mathcal{A}_\varepsilon^+ := \Bigl\{\omega\in\Omega : \sup_{t\in[0,T]} \bigl|\omega(t)\bigr| < \varepsilon\Bigr\}.$$

Clearly, $\mathcal{A}_\varepsilon^+ \in \mathcal{F}_0^T$ has positive probability, and thus, $\mathbb{P}(\mathcal{A} \cap \mathcal{A}_\varepsilon^+) = \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{A}_\varepsilon^+)$ is positive. Due to the compactness of $[0, T]$, there exists $\varepsilon > 0$ such that for all $\omega \in \mathcal{A}_\varepsilon^+$, we have

$$\left|\varphi(T, \omega, \delta/2) - \phi(T, \delta/2)\right| < \frac{\sqrt{\alpha}}{4} \quad \text{and} \quad \left|\varphi(T, \omega, -\delta/2) - \phi(T, -\delta/2)\right| < \frac{\sqrt{\alpha}}{4},$$

which implies together with (4.8) that

$$\varphi(T, \omega, \delta/2) > \frac{\sqrt{\alpha}}{4} \quad \text{and} \quad \varphi(T, \omega, -\delta/2) < -\frac{\sqrt{\alpha}}{4}.$$

Due to the fact that $|a_\alpha(\omega)| < \frac{\delta}{2}$ for all $\omega \in \mathcal{A} \cap \mathcal{A}_\varepsilon^+$, we get that for all $\omega \in \mathcal{A} \cap \mathcal{A}_\varepsilon^+$

$$\begin{aligned}
&\sup_{x \in (-\delta, \delta)} \left|\varphi\bigl(T, \omega, a_\alpha(\omega) + x\bigr) - a_\alpha(\theta_T \omega)\right| \\
&\quad \geq \max\Bigl\{\left|\varphi(T, \omega, \delta/2) - a_\alpha(\theta_T \omega)\right|, \left|\varphi(T, \omega, -\delta/2) - a_\alpha(\theta_T \omega)\right|\Bigr\}.
\end{aligned}$$

Consequently,

$$\sup_{x \in (-\delta, \delta)} \operatorname*{ess\,sup}_{\omega \in \Omega} \left|\varphi\bigl(t, \omega, a_\alpha(\omega) + x\bigr) - a_\alpha(\theta_t \omega)\right| > \frac{\sqrt{\alpha}}{4},$$

which contradicts (4.7) and the proof is complete. □

For the description of the bifurcation via finite-time properties, we consider a compact time interval $[0, T]$ and define the corresponding *finite-time Lyapunov exponent* associated with the invariant measure $\delta_{a_\alpha(\omega)}$ by

$$\lambda_\alpha^{T,\omega} := \frac{1}{T} \ln\left|\frac{\partial \varphi_\alpha}{\partial x}\bigl(T, \omega, a_\alpha(\omega)\bigr)\right|.$$

Clearly, the (classical) Lyapunov exponent λ_α^∞ associated with the random equilibrium $a_\alpha(\omega)$ is given by

$$\lambda_\alpha^\infty = \lim_{T \to \infty} \lambda_\alpha^{T,\omega}.$$

In contrast to the classical Lyapunov exponent, the finite-time Lyapunov exponent is, in general, a non-constant random variable.

Theorem 4.3 (Random pitchfork bifurcation, finite-time description). *Consider the stochastic differential equation* (4.1) *with the unique attracting random equilibrium* $\{a_\alpha(\omega)\}_{\omega \in \Omega}$. *For any finite time interval* $[0, T]$, *let* $\lambda_\alpha^{T,\omega}$ *denote the finite-time Lyapunov exponent associated with* $\{a_\alpha(\omega)\}_{\omega \in \Omega}$. *Then the following statements hold*:

(i) *For* $\alpha < 0$, *the random attractor* $\{a_\alpha(\omega)\}_{\omega \in \Omega}$ *is finite-time attractive*, *i.e.*

$$\lambda_\alpha^{T,\omega} \leq \alpha < 0 \quad \textit{for all } \omega \in \Omega.$$

(ii) *For* $\alpha > 0$, *the random attractor* $\{a_\alpha(\omega)\}_{\omega \in \Omega}$ *is not finite-time attractive*, *i.e.*

$$\mathbb{P}\bigl\{\omega \in \Omega : \lambda_\alpha^{T,\omega} > 0\bigr\} > 0.$$

Proof. (i) follows directly from Theorem 4.2(i).

(ii) Choose $\varepsilon := \frac{\sqrt{\alpha}}{2} > 0$. According to Proposition 4.1, there exists a measurable set $\mathcal{A} \in \mathcal{F}_{-\infty}^T$ of positive probability such that for all $\omega \in \mathcal{A}$

$$a_\alpha(\theta_s \omega) \in (-\varepsilon, \varepsilon) \quad \text{for all } s \in [0, T].$$

Let $\omega \in \mathcal{A}$ be arbitrary and we will estimate $\lambda_\alpha^{T,\omega}$. Let $\Phi_\alpha(t,\omega) := \frac{\partial \varphi_\alpha}{\partial x}(t,\omega,a_\alpha(\omega))$ denote the linearized random dynamical system along the random equilibrium $a_\alpha(\omega)$. Note that the linearized equation along the random equilibrium $a_\alpha(\omega)$ is given by

$$\dot{\xi}(t) = \big(\alpha - 3a_\alpha(\theta_t\omega)^2\big)\xi(t),$$

from which we derive that

$$\Phi_\alpha(t,\omega) = \exp\bigg(\int_0^t \big(\alpha - 3a_\alpha(\theta_s\omega)^2\big)\,\mathrm{d}s\bigg).$$

We thus get

$$\lambda_\alpha^{T,\omega} = \alpha - \frac{1}{T}\int_0^T 3a_\alpha(\theta_t\omega)^2\,\mathrm{d}t \geq \frac{\alpha}{4},$$

which completes the proof. □

Remark 4.4. Finite-time Lyapunov exponents are numerically computable quantities which measure expansion or contraction rates in a vicinity of a random equilibrium, and the change of the sign of finite-time Lyapunov exponents marks a bifurcation point observable from a practical point of view. For a system with some specified structures such as positivity, we refer to [24] for a powerful method to compute the maximal Lyapunov exponent with an explicit bound.

4.3. *The dichotomy spectrum at the bifurcation point*

We will compute the dichotomy spectrum of the linearization around the unique attracting random equilibrium $\{a_\alpha(\omega)\}$ of the system (4.1). As a direct consequence, we observe that hyperbolicity is lost at the bifurcation point $\alpha = 0$.

Theorem 4.5. *Let $\Phi_\alpha(t,\omega) := \frac{\partial \varphi_\alpha}{\partial x}(t,\omega,a_\alpha(\omega))$ denote the linearized random dynamical system along the random equilibrium $a_\alpha(\omega)$. Then the dichotomy spectrum Σ_α of Φ_α is given by*

$$\Sigma_\alpha = [-\infty, \alpha] \quad \textit{for all } \alpha \in \mathbb{R}.$$

Proof. From the linearized equation along $a_\alpha(\omega)$

$$\dot{\xi}(t) = \big(\alpha - 3a_\alpha(\theta_t\omega)^2\big)\xi(t),$$

we derive that

$$\Phi_\alpha(t,\omega) = \exp\bigg(\int_0^t \big(\alpha - 3a_\alpha(\theta_s\omega)^2\big)\,\mathrm{d}s\bigg). \tag{4.9}$$

Consequently,

$$\big|\Phi_\alpha(t,\omega)\big| \leq e^{\alpha|t|} \quad \text{for all } t \in \mathbb{R},$$

which implies that $\Sigma_\alpha \subset [-\infty, \alpha]$. Thus, it is sufficient to show that $[-\infty,\alpha] \subset \Sigma_\alpha$. For this purpose, let $\gamma \in (-\infty, \alpha]$ be arbitrary. Suppose the opposite, that Φ_α admits an exponential dichotomy with growth rate γ, invariant projector P_γ and positive constants K, ε. We now consider the two possible cases: (i) $P_\gamma = \mathbb{1}$ and (ii) $P_\gamma = 0$:

Case (i). $P_\gamma = \mathbb{1}$, i.e. we have

$$\Phi_\alpha(t,\omega) \leq K e^{(\gamma-\varepsilon)t} \quad \text{for all } t \geq 0. \tag{4.10}$$

Choose and fix $T > 0$ such that $e^{\frac{\varepsilon}{4}T} > K$. According to Proposition 4.1, there exists a measurable set $\mathcal{A} \in \mathcal{F}_{-\infty}^{T}$ of positive measure such that

$$a_\alpha(\theta_s\omega) \in (-\sqrt{\varepsilon}/2, \sqrt{\varepsilon}/2) \quad \text{for all } \omega \in \mathcal{A} \text{ and } s \in [0, T].$$

From (4.9) we derive that

$$\left|\Phi_\alpha(T, \omega)\right| \geq e^{T(\alpha - \frac{3\varepsilon}{4})} > Ke^{(\gamma-\varepsilon)T},$$

which is a contradiction to (4.10).

Case (ii): $P_\gamma = 0$, i.e. we have for almost all $\omega \in \Omega$

$$\Phi_\alpha(t, \theta_{-t}\omega) \geq \frac{1}{K}e^{(\gamma+\varepsilon)t} \quad \text{for all } t \geq 0,$$

which together with (4.9) implies that

$$\frac{\ln K + (\alpha - \gamma)t}{3} \geq \int_0^t a_\alpha(\theta_s\theta_{-t}\omega)^2\, \mathrm{d}s. \tag{4.11}$$

Choose and fix $T > 0$ such that

$$\frac{(T-1)^3}{3} > \frac{\ln K + (\alpha - \gamma)T}{3}.$$

Consider the following integral equation

$$x(t) = \int_0^t \left(\alpha x(s) - x(s)^3\right) \mathrm{d}s + \frac{t^4}{4} - \alpha\frac{t^2}{2} + t.$$

Clearly, the explicit solution of the above equation is $x(t) = t$. Due to the compactness of $[0, T]$, there exists an $\varepsilon > 0$ such that for any $x(0) \in (-\varepsilon, \varepsilon)$ and $\omega(t)$ with $\sup_{t\in[0,T]} |\omega(t) - \frac{t^4}{4} + \alpha\frac{t^2}{2} - t| \leq \varepsilon$ then the solution $x(t)$ of the following equation

$$x(t) = x(0) + \int_0^t \left(\alpha x(s) - x(s)^3\right) \mathrm{d}s + \omega(t)$$

satisfies that $\sup_{t\in[0,T]} |x(t) - t| \leq 1$. According to Proposition 4.1, there exists a measurable set $\mathcal{A}_\varepsilon^- \in \mathcal{F}_{-\infty}^0$ of positive measure such that $a_\alpha(\omega) \in (-\varepsilon, \varepsilon)$ for all $\omega \in \mathcal{A}_\varepsilon^-$. Define $\mathcal{A}_\varepsilon^+ \in \mathcal{F}_0^T$ by

$$\mathcal{A}_\varepsilon^+ := \left\{\omega \in \Omega : \sup_{t\in[0,T]} \left|\omega(t) - t^4/4 + \alpha t^2/2 - t\right| \leq \varepsilon\right\}.$$

Therefore, for all $\omega \in \mathcal{A}_\varepsilon^- \cap \mathcal{A}_\varepsilon^+$, we get

$$\sup_{t\in[0,T]} \left|a_\alpha(\theta_t\omega) - t\right| \leq 1,$$

which implies that

$$\int_0^T a_\alpha(\theta_s\omega)^2\, \mathrm{d}s > \frac{(T-1)^3}{3} > \frac{\ln K + (\alpha - \gamma)T}{3}.$$

Note that $\mathbb{P}(\mathcal{A}_\varepsilon^- \cap \mathcal{A}_\varepsilon^+) = \mathbb{P}(\mathcal{A}_\varepsilon^-)\mathbb{P}(\mathcal{A}_\varepsilon^+) > 0$. Then for $\omega \in \theta_T(\mathcal{A}_\varepsilon^- \cap \mathcal{A}_\varepsilon^+)$, the above leads to a contradiction to (4.11), and the proof is complete. $\square$

We have seen in Theorem 4.3 that the bifurcation of (4.1) manifests itself also via finite-time Lyapunov exponents: before the bifurcation, all finite-time Lyapunov exponents are negative, and after the bifurcation, one observes positive finite-time Lyapunov exponents with positive probability for arbitrarily large times. This implies in particular that for positive α the set of all finite-time Lyapunov exponents observed on a set of full measure does not converge to the (classical) Lyapunov exponent when time tends to infinity. The following theorem makes precise the fact that, in contrast to the Lyapunov spectrum, the dichotomy spectrum includes limits of the set of finite-time Lyapunov exponents.

Theorem 4.6. *Let (θ, Φ) be a linear random dynamical system on $\mathbb{R}^d$ with dichotomy spectrum Σ. Define the finite-time Lyapunov exponent*

$$\lambda(T, \omega, x) := \frac{1}{T} \ln \frac{\|\Phi(T, \omega)x\|}{\|x\|} \quad \textit{for all } T > 0, \omega \in \Omega \textit{ and } x \in \mathbb{R}^d \setminus \{0\}.$$

Then

$$\lim_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T, \omega, x) = \sup \Sigma$$

provided that $\sup \Sigma < \infty$ *and*

$$\lim_{T\to\infty} \operatorname*{ess\,inf}_{\omega\in\Omega} \inf_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T, \omega, x) = \inf \Sigma$$

provided that $\inf \Sigma > -\infty$.

Proof. By definition of $\lambda(T, \omega, x)$, we get that for all $T, S \geq 0$

$$(T+S) \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T+S, \omega, x) \leq T \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T, \omega, x) + S \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(S, \omega, x).$$

This implies that the function $[0, \infty) \cap \mathbb{T} \to \mathbb{R}, T \mapsto T \operatorname{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T, \omega, x)$ is subadditive; we thus obtain that the limit $T \to \infty$ exists and so

$$\lim_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T, \omega, x) = \limsup_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T, \omega, x).$$

We first prove that provided $\sup \Sigma < \infty$, we have

$$\gamma := \limsup_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T, \omega, x) = \sup \Sigma.$$

Since $\sup \Sigma < \infty$ it follows that there exists $K > 0$ such that

$$\big\|\Phi(t, \omega)\big\| \leq K e^{t \sup \Sigma} \quad \text{for all } t \geq 0. \tag{4.12}$$

Assume first that $\gamma < \sup \Sigma$. This means that there exists a $t_0 > 0$ such that for all $t \geq t_0$ and for almost all $\omega \in \Omega$, we have $\|\Phi(t, \omega)\| \leq e^{t(\gamma+\sup \Sigma)/2}$. Thus, together with (4.12), we obtain for all $t \geq 0$ and for almost all $\omega \in \Omega$ that

$$\big\|\Phi(t, \omega)\big\| \leq \widehat{K} e^{t(\gamma+\sup \Sigma)/2}, \qquad \widehat{K} := \max\big\{1, K e^{t_0(\sup \Sigma - \gamma)/2}\big\}.$$

Hence, $\sup \Sigma \leq (\gamma + \sup \Sigma)/2$, which is a contradiction. Assume now that $\gamma > \sup \Sigma$. This means in particular that $\sup \Sigma < \infty$. Hence, there exists a $K > 0$ such that for almost all $\omega \in \Omega$, we have

$$\big\|\Phi(t, \omega)x\big\| \leq K e^{t(\gamma+\sup \Sigma)/2} \|x\| \quad \text{for all } x \in \mathbb{R}^d.$$

This leads to $\lambda(t,\omega,x) \le (\gamma + \sup\Sigma)/2$ for all $x \in \mathbb{R}^d \setminus \{0\}$, and thus,

$$\gamma = \limsup_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T,\omega,x) \le (\gamma + \sup\Sigma)/2,$$

which proves the first equality. Similarly, one can show that

$$\lim_{T\to\infty} \operatorname*{ess\,inf}_{\omega\in\Omega} \inf_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T,\omega,x) = \inf\Sigma$$

provided that $\inf\Sigma > -\infty$, which finishes the proof of this theorem. □

In the following example, we construct explicitly a linear random dynamical system with $\sup\Sigma = \infty$ but

$$\lim_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T,\omega,x) < \infty.$$

An example of a linear random dynamical system with $\inf\Sigma = -\infty$ but

$$\lim_{T\to\infty} \operatorname*{ess\,inf}_{\omega\in\Omega} \inf_{x\in\mathbb{R}^d\setminus\{0\}} \lambda(T,\omega,x) > -\infty$$

can be constructed analogously. This example shows the importance of the assumption $\sup\Sigma < \infty$ or $\inf\Sigma > -\infty$ in the above theorem.

Example 4.7. Following the construction in Example 3.8, there exist infinitely many measurable sets $\{U_n\}_{n\in\mathbb{N}}$ of positive measure such that for $n \ge 2$, $U_n, \theta U_n, \theta^2 U_n$ are pairwise disjoint. We define a random mapping $A : \Omega \to \mathbb{R}$ as follows:

$$A(\omega) = \begin{cases} \frac{1}{n}, & \omega \in U_n \cup \theta^2 U_n, n \ge 2, \\ n, & \omega \in \theta U_n, n \ge 2, \\ 1, & \text{otherwise}. \end{cases}$$

Let Φ denote the discrete-time RDS generated by A. Since $\ln\|A(\cdot)\|$ is neither bounded from above nor from below, we get that $\Sigma(\Phi) = [-\infty, \infty]$. On the other hand, it is easy to see that for all $T \ge 2$ we get that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \bigl|\Phi(T,\omega)\bigr| = 1,$$

which implies that

$$\lim_{T\to\infty} \operatorname*{ess\,sup}_{\omega\in\Omega} \frac{1}{T} \ln\bigl|\Phi(T,\omega)\bigr| = 0.$$

5. Topological equivalence of random dynamical systems

This section deals with topological equivalence of random dynamical systems [1,16,17,21]. This concept has not been used so far to study bifurcations of random dynamical systems, and the main aim of this section is to discuss topological equivalence for the stochastic differential equation (4.1) from Section 4, given by

$$\mathrm{d}x = \bigl(\alpha x - x^3\bigr)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t.$$

The concept of topological equivalence for random dynamical systems [1, Definition 9.2.1] differs from the corresponding deterministic notion of topological equivalence in the sense that instead of one homeomorphism (mapping orbits to orbits), the random version is given by a family of homeomorphisms $\{h_\omega\}_{\omega\in\Omega}$. The precise definition for one-sided random dynamical systems is given as follows.

Definition 5.1 (Topological equivalence). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\theta : \mathbb{T} \times \Omega \to \Omega$ a metric dynamical system and (X_1, d_1), (X_2, d_2) be metric spaces. Then two one-sided random dynamical systems $(\varphi_1 : \mathbb{T}_0^+ \times \Omega \times X_1 \to X_1, \theta)$ and $(\varphi_2 : \mathbb{T}_0^+ \times \Omega \times X_2 \to X_2, \theta)$ are called *topologically equivalent* if there exists a conjugacy $h : \Omega \times X_1 \to X_2$ fulfilling the following properties:

(i) For almost all $\omega \in \Omega$, the function $x \mapsto h(\omega, x)$ is a homeomorphism from X_1 to X_2.
(ii) The mappings $(\omega, x_1) \mapsto h(\omega, x_1)$ and $(\omega, x_2) \mapsto h^{-1}(\omega, x_2)$ are measurable.
(iii) The random dynamical systems φ_1 and φ_2 are *cohomologous*, i.e.

$$\varphi_2\big(t, \omega, h(\omega, x)\big) = h\big(\theta_t\omega, \varphi_1(t, \omega, x)\big) \quad \text{for all } t \geq 0, x \in X_1 \text{ and almost all } \omega \in \Omega.$$

A bifurcation is then described by means of a lack of topological equivalence at the bifurcation point. The following theorem says that near the bifurcation point $\alpha = 0$, all systems of (4.1) are equivalent.

Theorem 5.2. *Let φ_α denote the one-sided RDS generated by the SDE (4.1). Then there exists an $\varepsilon > 0$ such that for all $\alpha \in (-\varepsilon, \varepsilon)$ the random dynamical systems φ_α are topologically equivalent to the dynamical system $(e^{-t}x)_{t,x\in\mathbb{R}}$.*

Proof. Let $a_\alpha(\omega)$ denote the unique random equilibrium of (4.1). According to the results in [9], we obtain that

$$\mathbb{E}a_\alpha(\omega)^2 = \frac{\int_{-\infty}^{\infty} u^2 \exp(\frac{1}{\sigma^2}(\alpha u^2 - \frac{1}{2}u^4))\,\mathrm{d}u}{\int_{-\infty}^{\infty} \exp(\frac{1}{\sigma^2}(\alpha u^2 - \frac{1}{2}u^4))\,\mathrm{d}u},$$

and therefore,

$$\lim_{\alpha\to 0} \mathbb{E}a_\alpha(\omega)^2 = \frac{\int_{-\infty}^{\infty} u^2 \exp(-\frac{u^4}{2\sigma^2})\,\mathrm{d}u}{\int_{-\infty}^{\infty} \exp(-\frac{u^4}{2\sigma^2})\,\mathrm{d}u} > 0.$$

Then there exists an $\varepsilon > 0$ such that for all $\alpha \in (-\varepsilon, \varepsilon)$, we have

$$\delta := \frac{3}{4}\mathbb{E}a_\alpha(\omega)^2 - \alpha > 0.$$

For any $x \in \mathbb{R}$ and $(t, \omega) \in \mathbb{R}_0^+ \times \Omega$, we define

$$\psi(t, \omega, x) := \varphi_\alpha\big(t, \omega, x + a_\alpha(\omega)\big) - a_\alpha(\theta_t\omega). \tag{5.1}$$

By using the transformation function $f(\omega, x) := x - a_\alpha(\omega)$, the random dynamical systems φ_α and ψ are topologically equivalent. Hence, it is sufficient to show that ψ is topologically equivalent to the dynamical system $(e^{-t}x)_{t,x\in\mathbb{R}}$; the proof of this is divided into four parts.

Part 1. We first summarise some properties of ψ:

(1) Since $a_\alpha(\omega)$ is a random equilibrium of φ_α, it follows that

$$\psi(t, \omega, 0) = 0 \quad \text{for all } t \geq 0 \text{ and } \omega \in \Omega. \tag{5.2}$$

(2) Due to the monotonicity of φ_α, for $x_1 > x_2$, we have

$$\psi(t, \omega, x_1) > \psi(t, \omega, x_2) \quad \text{for all } \omega \in \Omega \text{ and } t \geq 0. \tag{5.3}$$

(3) From (4.1), we derive that

$$\begin{aligned}\psi(t, \omega, x) = x + \int_0^t \psi(s, \omega, x)\big(&\alpha - a_\alpha(\theta_s\omega)^2 - a_\alpha(\theta_s\omega)\varphi_\alpha\big(s, \omega, a_\alpha(\omega) + x\big) \\ &- \varphi_\alpha\big(s, \omega, a_\alpha(\omega) + x\big)^2\big)\,\mathrm{d}s,\end{aligned}$$

consequently,

$$\psi(t,\omega,x) = x\exp\biggl(\int_0^t \alpha - a_\alpha(\theta_s\omega)^2 - a_\alpha(\theta_s\omega)\varphi_\alpha\bigl(s,\omega,a_\alpha(\omega)+x\bigr) - \varphi_\alpha\bigl(s,\omega,a_\alpha(\omega)+x\bigr)^2\,\mathrm{d}s\biggr). \tag{5.4}$$

Part 2. We shall now demonstrate some estimates on ψ. According to Birkhoff's Ergodic Theorem, there exists an invariant set $\widetilde{\Omega}$ of full measure such that

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t a_\alpha(\theta_s\omega)^2\,\mathrm{d}s = \mathbb{E}a_\alpha(\omega)^2. \tag{5.5}$$

Choose and fix $\omega\in\widetilde{\Omega}$. From (5.5), there exists $T>0$ such that for all $|t|>T$ we have

$$\left|\frac{1}{t}\int_0^t a_\alpha(\theta_s\omega)^2\,\mathrm{d}s - \mathbb{E}a_\alpha(\omega)^2\right| \le \delta. \tag{5.6}$$

The elementary inequality $u^2+uv+v^2 \ge \frac{3}{4}u^2$ for $u,v\in\mathbb{R}$ implies with (5.4) that for $x>0$

$$\psi(t,\omega,x) \le x\exp\biggl(\int_0^t \alpha - \frac{3}{4}a_\alpha(\theta_s\omega)^2\,\mathrm{d}s\biggr),$$

then for $t\ge T$, (5.6) implies the following estimate

$$\psi(t,\omega,x) \le xe^{-\frac{\delta}{4}t}, \quad \text{for all } x>0. \tag{5.7}$$

Note that the function ψ as defined in (5.1) can be defined for certain negative times also (as the cocycle φ_α is invertible locally), and we have

$$\lim_{t\to\tilde{\kappa}(\omega,x)+}\psi(t,\omega,x) = \infty \quad \text{for } x>0 \quad \text{and} \quad \lim_{t\to\tilde{\kappa}(\omega,x)+}\psi(t,\omega,x) = -\infty \quad \text{for } x<0, \tag{5.8}$$

where $\tilde{\kappa}(\omega,x)$ denotes the infimum of the domain of the function $t\mapsto\psi(t,\omega,x)$.

Part 3. We now show the required conjugacy. By (5.2) and (5.3), for $x>0$ we have $\psi(s,\omega,x)>0$ for all $s\ge 0$, and consequently by (5.7) and (5.8) we obtain that

$$\lim_{r\to\infty}\int_r^\infty \psi(s,\omega,x)\,\mathrm{d}s = 0 \quad \text{and} \quad \lim_{r\to\tilde{\kappa}(\omega,x)+}\int_r^\infty \psi(s,\omega,x)\,\mathrm{d}s = \infty.$$

Hence there exists a unique $r(\omega,x)$ such that

$$\int_{r(\omega,x)}^\infty \psi(s,\omega,x)\,\mathrm{d}s = 1. \tag{5.9}$$

Similarly, $r(\omega,x)$ for $x<0$ is defined to satisfy

$$\int_{r(\omega,x)}^\infty \psi(s,\omega,x)\,\mathrm{d}s = -1, \tag{5.10}$$

and we define $r(\omega,0) := -\infty$. Using the cocycle property of ψ, we obtain that

$$r(\omega,x) = r\bigl(\theta_s\omega,\psi(s,\omega,x)\bigr) + s. \tag{5.11}$$

Define the function

$$g(\omega,x) := \begin{cases} e^{r(\omega,x)}, & x>0,\\ 0, & x=0,\\ -e^{r(\omega,x)}, & x<0. \end{cases}$$

We will now show that g transforms the random dynamical system ψ to the dynamical system $(e^{-t}x)_{t,x\in\mathbb{R}}$. For any $x>0$, we have $\psi(s,\omega,x)>0$ and thus from the definition of the function g it follows that

$$g\big(\theta_s\omega,\psi(s,\omega,x)\big) = e^{r(\theta_s\omega,\psi(s,\omega,x))},$$

which implies together with (5.11) that

$$g\big(\theta_s\omega,\psi(s,\omega,x)\big) = e^{r(\omega,x)-s} = e^{-s}g(\omega,x).$$

Similarly, for $x<0$ we also have $g(\theta_s\omega,\psi(s,\omega,x)) = e^{-s}g(\omega,x)$ for all $s\in(\tilde\kappa(\omega,x),\infty)$, $\omega\in\Omega$.

Part 4. We will show that $g_\omega:\mathbb{R}\to\mathbb{R}$, $x\mapsto g(\omega,x)$ is a homeomorphism, and that g is jointly measurable. Choose and fix $\omega\in\widetilde\Omega$.

Injectivity: From the definition of g, it is easily seen that for $x_1>0>x_2$ we have

$$g_\omega(x_1)>0>g_\omega(x_2).$$

On the other hand, based on strict monotonicity of ψ we get that for $x_1>x_2>0$

$$\int_{r(\omega,x_2)}^\infty \psi(s,\omega,x_1)\,\mathrm{d}s > \int_{r(\omega,x_2)}^\infty \psi(s,\omega,x_2)\,\mathrm{d}s = 1.$$

Consequently, $r(\omega,x_1)>r(\omega,x_2)$ and thus $g_\omega(x_1)>g_\omega(x_2)$. Similarly, for $0>x_1>x_2$ we also have $g_\omega(x_1)>g_\omega(x_2)$. Therefore, g_ω is strictly increasing and thus injective.

Continuity: We first show that $\lim_{x\to0+}g_\omega(x)=0$. Let $\varepsilon>0$ be arbitrary. Choose $\tilde T>T$ such that $\frac{4}{\delta}e^{-\frac{\delta}{4}\tilde T}<\frac13$ and $e^{-\tilde T}<\varepsilon$. By (5.7), for all $t\ge\tilde T$ we have

$$\psi(t,\omega,x)\le e^{-\frac{\delta}{4}t}x.$$

As a consequence, for all $x\in(0,1)$ we get

$$\int_{\tilde T}^\infty \psi(s,\omega,x)\,\mathrm{d}s \le \int_{\tilde T}^\infty e^{-\frac{\delta}{4}s}\,\mathrm{d}s < \frac13. \tag{5.12}$$

Since $\lim_{x\to0}\psi(s,\omega,x)=0$, $[-\tilde T,\tilde T]$ is a compact interval and $\lim_{x\to0}\tilde\kappa(\omega,x)=-\infty$, there exists $\delta^*>0$ such that

$$\int_{-\tilde T}^{\tilde T}\psi\big(s,\omega,\delta^*\big)\,\mathrm{d}s<\frac13,$$

which together with (5.12) implies that

$$\int_{-\tilde T}^\infty \psi(s,\omega,x)\,\mathrm{d}s<\frac23 \quad \text{for all } x\in\big(0,\min\{1,\delta^*\}\big).$$

Therefore, $r(\omega,x)<-\tilde T$ and thus $g_\omega(x)<\varepsilon$ for all $x\in(0,\min\{1,\delta^*\})$. Hence, $\lim_{x\to0+}g_\omega(x)=0$. One can similarly show that $\lim_{x\to0-}g_\omega(x)=0$, and thus g_ω is continuous at 0. The continuity of g on the whole real line can be proved in a similar way.

Surjectivity: It is easy to prove surjectivity from

$$\lim_{x\to\infty}g_\omega(x)=\infty \quad\text{and}\quad \lim_{x\to-\infty}g_\omega(x)=-\infty.$$

Measurability: By the definition of g, in order to prove the joint measurability of g it is enough to show the joint measurability of the mapping $(\omega, x) \mapsto r(\omega, x)$. Since the map $x \mapsto r(\omega, x)$ is continuous for each fixed $\omega \in \Omega$, it follows from e.g. [7, Lemma 1.1] that it is sufficient to show that the map $\omega \mapsto r(\omega, x)$ is measurable for each fixed $x \in \mathbb{R}$. Choose and fix an arbitrary $x > 0$, and let $\beta \in \mathbb{R}$ be arbitrary. Then, by the definition of $r(\omega, x)$ we have

$$\begin{aligned}\{\omega : r(\omega, x) \le \beta\} &= \left\{\omega : \int_\beta^\infty \psi(t, \omega, x)\,\mathrm{d}t \le 1\right\}\\ &= \bigcap_{n\in\mathbb{N}, n\ge\beta} \left\{\omega : \int_\beta^n \psi(t, \omega, x)\,\mathrm{d}t < 1\right\}.\end{aligned}$$

It should be clear that for each $n \in \mathbb{N}$, the map $\omega \mapsto \int_\beta^n \psi(t, \omega, x)\,\mathrm{d}t$ is measurable, and consequently the map $\omega \mapsto r(\omega, x)$ is measurable. The case $x < 0$ is similar, and we have defined $r(\omega, 0) = -\infty$ for all $\omega \in \Omega$. Thus we obtain the measurability of the map $\omega \mapsto r(\omega, x)$ for all $x \in \mathbb{R}$.

This completes the proof of this theorem. □

This theorem implies that the stochastic differential equation (4.1) does not admit a bifurcation at $\alpha = 0$ which is induced by the above concept of topological equivalence. In addition, because of the observations in Theorem 4.5, this concept of equivalence is not in correspondence with the dichotomy spectrum (linear systems which are hyperbolic and non-hyperbolic can be equivalent).

We will show now that the concept of a *uniform topological equivalence* is the right tool to obtain the bifurcations studied in this paper.

Definition 5.3 (Uniform topological equivalence). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\theta : \mathbb{T} \times \Omega \to \Omega$ a metric dynamical system and (X_1, d_1), (X_2, d_2) be metric spaces. Then two one-sided random dynamical systems $(\varphi_1 : \mathbb{T}_0^+ \times \Omega \times X_1 \to X_1, \theta)$ and $(\varphi_2 : \mathbb{T}_0^+ \times \Omega \times X_2 \to X_2, \theta)$ are called *uniformly topologically equivalent* with respect to a random equilibrium $\{a(\omega)\}_{\omega\in\Omega}$ of φ_1 if there exists a conjugacy $h : \Omega \times X_1 \to X_2$ fulfilling the following properties:

(i) For almost all $\omega \in \Omega$, the function $x \mapsto h(\omega, x)$ is a homeomorphism from X_1 to X_2.
(ii) The mappings $(\omega, x_1) \mapsto h(\omega, x_1)$ and $(\omega, x_2) \mapsto h^{-1}(\omega, x_2)$ are measurable.
(iii) The random dynamical systems φ_1 and φ_2 are *cohomologous*, i.e.

$$\varphi_2\big(t, \omega, h(\omega, x)\big) = h\big(\theta_t\omega, \varphi_1(t, \omega, x)\big) \quad \text{for all } t \ge 0, x \in X_1 \text{ and almost all } \omega \in \Omega.$$

(iv) We have

$$\lim_{\delta\to 0} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_2\big(h(\omega, x), h\big(\omega, a(\omega)\big)\big) = 0$$

and

$$\lim_{\delta\to 0} \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(h(\omega,a(\omega)))} d_1\big(h^{-1}(\omega, x), a(\omega)\big) = 0.$$

Note that, in comparison to the concept of topological equivalence (Definition 5.1), we added (iv) to take uniformity into account.

We show now that uniform topological equivalence preserves local uniform attractivity.

Proposition 5.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\theta : \mathbb{T} \times \Omega \to \Omega$ a metric dynamical system and (X_1, d_1), (X_2, d_2) be metric spaces, and let $(\varphi_1 : \mathbb{T}_0^+ \times \Omega \times X_1 \to X_1, \theta)$ and $(\varphi_2 : \mathbb{T}_0^+ \times \Omega \times X_2 \to X_2, \theta)$ be two one-sided random dynamical systems which are* uniformly topologically equivalent *with respect to a random equilibrium $\{a(\omega)\}_{\omega\in\Omega}$ of φ_1. Let $h : \Omega \times X_1 \to X_2$ denote the conjugacy. Then $\{a(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_1 if and only if $\{h(\omega, a(\omega))\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_2.*

Proof. Suppose that $\{a(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_1 and let $\eta > 0$. Then there exists a $\gamma > 0$ such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\gamma(a(\omega))} d_2\big(h(\omega, x), h\big(\omega, a(\omega)\big)\big) \le \eta.$$

Since $\{a(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_1, there exists a $\delta > 0$ and a $T > 0$ such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_1\big(\varphi_1(t, \omega, x), a(\theta_t\omega)\big) \le \frac{\gamma}{2} \quad \text{for all } t \ge T.$$

Hence, for all $t \ge T$, we have

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_2\big(h\big(\theta_t\omega, \varphi_1(t, \omega, x)\big), h\big(\theta_t\omega, a(\theta_t\omega)\big)\big) \le \eta.$$

This means that for all $t \ge T$, we have

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_2\big(\varphi_2\big(t, \omega, h(\omega, x)\big), h\big(\theta_t\omega, a(\theta_t\omega)\big)\big) \le \eta,$$

and there exists a $\beta > 0$ such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\beta(h(\omega,a(\omega)))} d_1\big(h^{-1}(\omega, x), a(\omega)\big) \le \frac{\delta}{2}.$$

Finally, this means that for all $t \ge T$, we have

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\beta(h(\omega,a(\omega)))} d_2\big(\varphi_2(t, \omega, x), h\big(\theta_t\omega, a(\theta_t\omega)\big)\big) \le \eta,$$

which finishes the proof that $\{h(\omega, a(\omega))\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_2; the converse is proved similarly. □

As a consequence of this proposition, it follows that (4.1) admits a bifurcation.

Theorem 5.5. *The stochastic differential equation* (4.1) *admits a random bifurcation at* $\alpha = 0$ *which is induced by the concept of uniform topological equivalence.*

Proof. This is a direct consequence of Theorem 4.2 and Proposition 5.4. □

Appendix

A.1. *Metric dynamical systems*

Let $\mathcal{B}(Y)$ denote the Borel σ-algebra of a metric space Y. Consider a time set $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}, \mathcal{F})$-measurable function $\theta : \mathbb{T} \times \Omega \to \Omega$ is called a *measurable dynamical system* if $\theta(0, \omega) = \omega$ and $\theta(t + s, \omega) = \theta(t, \theta(s, \omega))$ for all $t, s \in \mathbb{T}$ and $\omega \in \Omega$. We use the abbreviation $\theta_t\omega$ for $\theta(t, \omega)$. A measurable dynamical system is said to be *measure preserving* or *metric* if $\mathbb{P}\theta(t, A) = \mathbb{P}A$ for all $t \in \mathbb{T}$ and $A \in \mathcal{F}$, and such a dynamical system is called *ergodic* if for any $A \in \mathcal{F}$ satisfying $\theta_t A = A$ for all $t \in \mathbb{T}$, one has $\mathbb{P}A \in \{0, 1\}$. A particular metric dynamical system, which naturally is used when dealing with (one-dimensional) stochastic differential equations, is generated by the Brownian motion. More precisely, $\Omega := C_0(\mathbb{R}, \mathbb{R}) := \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$. Let Ω be equipped with the compact-open topology and the Borel σ-algebra $\mathcal{F} := \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}))$. Let $\mathbb{P}$ denote the Wiener probability measure on $(\Omega, \mathcal{F})$. The metric dynamical system is then given by the Wiener shift $\theta : \mathbb{R} \times \Omega \to \Omega$, defined by $\theta(t, \omega(\cdot)) := \omega(\cdot + t) - \omega(t)$, and it is well-known that θ is ergodic [1]. On $(\Omega, \mathcal{F})$, we have the natural filtration

$$\mathcal{F}_s^t := \sigma\big(\omega(u) - \omega(v) : s \leq u, v \leq t\big) \quad \text{for all } s \leq t,$$

with $\theta_u^{-1} \mathcal{F}_s^t = \mathcal{F}_{s+u}^{t+u}$.

A.2. *Invariant measures*

For a given random dynamical system (θ, φ), let $\Theta : \mathbb{T} \times \Omega \times X \to \Omega \times X$ denote the corresponding *skew product flow*, given by $\Theta(t, \omega, x) := (\theta_t \omega, \varphi(t, \omega)x)$. This is a measurable dynamical system on the extended phase space $\Omega \times X$. A probability measure μ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ is said to be an *invariant measure* if

(i) $\mu(\Theta_t A) = \mu(A)$ for all $t \in \mathbb{T}$ and $A \in \mathcal{F} \otimes \mathcal{B}$,
(ii) $\pi_\Omega \mu = \mathbb{P}$,

where $\pi_\Omega \mu$ denotes the marginal of μ on $(\Omega, \mathcal{F})$. If the metric space X is separable and complete, then an invariant measure μ admits a $\mathbb{P}$-almost surely unique *disintegration* [1, Proposition 1.4.3], that is a family of probability measures $(\mu_\omega)_{\omega \in \Omega}$ with

$$\mu(A) = \int_\Omega \int_X \mathbb{1}_A(\omega, x) \, \mathrm{d}\mu_\omega(x) \, \mathrm{d}\mathbb{P}(\omega).$$

A.3. *Random sets*

A function $\omega \mapsto M(\omega)$ taking values in the subsets of the phase space X of a random dynamical system is called a *closed* (*compact*, respectively) *random set* if $M(\omega)$ is closed (compact, respectively) for all $\omega \in \Omega$ and the map $\omega \mapsto d(x, M(\omega))$ is measurable for each $x \in X$. We use the term ω-fiber of M for the set $M(\omega)$. Note that if X is a Polish space, then closed (compact, respectively) random sets are measurable sets of the product σ-algebra $\mathcal{F} \otimes \mathcal{B}$ [11, Remark after Definition 14]. A random set M is called *invariant* with respect to the random dynamical system (θ, φ) if $\varphi(t, \omega)M(\omega) = M(\theta_t \omega)$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$.

A.4. *Random attractors*

A nonempty, compact and invariant random set $\omega \mapsto A(\omega)$ is called *global random attractor* [11,20] for a random dynamical system (θ, φ) with metric state space (X, d), if it attracts all bounded sets in the sense of pullback attraction, i.e., for all bounded sets $B \subset X$, one has

$$\lim_{t \to \infty} \operatorname{dist}\big(\varphi(t, \theta_{-t}\omega)B, A(\omega)\big) = 0 \quad \text{for almost all } \omega \in \Omega,$$

where $\operatorname{dist}(C, D) := \sup_{c \in C} d(c, D)$ is the *Hausdorff semi-distance* of C and D. A global random attractor (given it exists) is always unique [8]. The existence of random attractors is proved via so-called absorbing sets [12]. A compact random set $B(\omega)$ is called a *compact random absorbing set* if for almost all $\omega \in \Omega$ and any bounded set $D \subset X$, there exists a time $T > 0$ such that

$$\varphi(t, \theta_{-t}\omega)D \subset B(\omega) \quad \text{for all } t \geq T.$$

Suppose that $\phi(t, \omega, \cdot)$, $t \in \mathbb{T}$, $\omega \in \Omega$, is continuous. Given a compact random absorbing set $B(\omega)$, it follows that there exists a global random attractor $\{A(\omega)\}_{\omega \in \Omega}$, given by

$$A(\omega) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega)B(\omega)} \quad \text{for almost all } \omega \in \Omega.$$

A.5. Lyapunov exponents and multiplicative ergodic theory

Given a linear random dynamical system (θ, Φ) in $\mathbb{R}^d$, a *Lyapunov exponent* is given by

$$\lambda = \lim_{t \to \pm\infty} \frac{1}{|t|} \ln \left\| \Phi(t, \omega) x \right\| \quad \text{for some } \omega \in \Omega \text{ and } x \in \mathbb{R}^d \setminus \{0\}.$$

The Multiplicative Ergodic Theorem [1,22] shows that there are only finitely many Lyapunov exponents provided the random dynamical system is ergodic and fulfills an integrability condition. More precisely, consider a linear random dynamical system $(\theta : \mathbb{T} \times \Omega \to \Omega, \Phi : \mathbb{T} \times \Omega \to \mathbb{R}^{d \times d})$, suppose that θ is ergodic and Φ satisfies the integrability condition

$$\sup_{t \in [0,1]} \ln^+ \left(\left\| \Phi(t, \cdot)^{\pm 1} \right\| \right) \in L^1(\mathbb{P}),$$

here $\ln^+(x) := \max\{0, \ln(x)\}$. Then the Multiplicative Ergodic Theorem states that almost surely, there exist at most d Lyapunov exponents $\lambda_1 < \lambda_2 < \cdots < \lambda_p$ and fiber-wise decomposition

$$\mathbb{R}^d = O_1(\omega) \oplus O_2(\omega) \oplus \cdots \oplus O_p(\omega) \quad \text{for almost all } \omega \in \Omega$$

into Oseledets subspaces $O_i \subset \mathbb{R}^d$ such that for all $i \in \{1, \ldots, p\}$ and almost all $\omega \in \Omega$, one has

$$\lim_{t \to \pm\infty} \frac{1}{|t|} \ln \left\| \Phi(t, \omega) x \right\| = \lambda_i \quad \text{for all } x \in O_i(\omega) \setminus \{0\}.$$

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